Curvature Estimates for Graphical Mean Curvature Flow in Higher Codimension

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Zusammenfassung

Ich betrachte Deformationen von Abbildungen $f : M \to N$ unter dem mittleren Krümmungsfluss, d. h. den mittleren Krümmungsfluss des Graphen $\Gamma(f)$ von f in $M \times N$.

Sei M eine kompakte und N eine vollständige Riemannsche Fläche, deren Krümmungen für ein $\sigma > 0$ die Beziehung

$$\sec_N \leq \sigma \leq \sec_M$$

erfüllen. Falls die Startabbildung strikt flächenverkleinernd ist, so beweise ich zeitliche Abschätzungen für den mittleren Krümmungsvektor. Unter einer stärkeren Annahme an das Differential der Abbildung zeige ich eine zeitliche Abschätzung für die zweite Fundamentalform. Die Abschätzungen folgen aus dem Maximumprinzip und hängen von den Werten des Differentials der Startabbildung und den Krümmungen von M und N ab. Die Beweise basieren auf den in [STW14] entwickelten Ideen, wobei die zusätzlich auftretenden Krümmungsterme eine Schwierigkeit darstellen. Daher leite ich zuerst explizite Abschätzungen der Singulärwerte des Differentials der Abbildung her, durch die sich die Krümmungsterme kontrollieren lassen.

Für Abbildungen zwischen Euklidischen Räumen $f : \mathbb{R}^m \to \mathbb{R}^n$ zeige ich, dass es im Falle von Lipschitz-stetigen Anfangsdaten, die eine gewisse Schranke besitzen, eine Lösung des mittleren Krümmungsflusses gibt, die für alle Zeiten existiert und graphisch bleibt. Falls die Anfangsdaten im Unendlichen gegen Null gehen, so strebt die Lösung (gleichmäßig bezüglich der räumlichen Koordinaten) gegen Null. Der Fluss von Lagrangeschen Abbildungen zwischen Euklidischen Räumen wurde im nicht-kompakten Fall in [CCH12] und im kompakten Fall in [Smo04] behandelt. Nach einer Idee aus [SS14b] interpretiere ich die dort betrachtete Größe $\mathrm{s}_{\mathbb{R}^n \times \mathbb{R}^n}(\mathrm{J}\mathrm{d}F(\cdot), \mathrm{J}\mathrm{d}F(\cdot))$ als Tensor $-\mathrm{s}^{\perp}$ auf dem Normalenbündel der Untermannigfaltigkeit. Dieser Tensor existiert auch im nicht-Lagrangeschen Fall, so dass ich eine Erweiterung der dortigen Resultate erhalte.

Schlüsselworte: graphischer mittlerer Krümmungsfluss
vollständige Mannigfaltigkeiten
nicht-kompakter Euklidischer Raum

2010 Mathematics Subject Classification: 53C44, 53C42, 53A07

Abstract

I consider deformations of maps $f: M \to N$ under the mean curvature flow, i. e. the mean curvature flow of the graph $\Gamma(f)$ of f in $M \times N$.

Let M be a compact and N be a complete Riemann surface, such that for some fixed $\sigma > 0$ the curvatures satisfy the relation

$$\sec_N \leq \sigma \leq \sec_M \ .$$

If the initial map is strictly area-decreasing, I establish decay estimates for the mean curvature vector with respect to time. Also, assuming a stronger condition on the differential of the initial map, I show a decay estimate for the second fundamental form. These estimates follow from the maximum principle and depend on the values of the differential of the initial map as well as on the curvatures of M and N. The proofs are based on the ideas developed in [STW14], where here the additional curvature terms pose difficulties. To overcome these, I derive explicit estimates for the singular values of the differential of the evolving map, which then are used to control the curvature terms.

For maps between Euclidean spaces $f: \mathbb{R}^m \to \mathbb{R}^n$, I show that for Lipschitz continuous initial data satisfying a certain bound, there exists a solution of the mean curvature flow which exists for all times and remains graphic. Further, if the initial map tends to zero at spatial infinity, then also the solution tends to zero uniformly with respect to the spatial coordinates. The case of Lagrangian maps in the non-compact setting was considered in [CCH12] and in the compact case in [Smo04]. Following an idea from [SS14b], I reinterpret the tensor $\mathrm{s}_{\mathbb{R}^n \times \mathbb{R}^n}(\mathrm{J}\cdot, \mathrm{J}\cdot)$ considered in the Lagrangian case as a tensor $-\mathrm{s}^{\perp}$ on the normal bundle of the submanifold. This tensor always exists and allows to extend the results to the non-Lagrangian case.

Keywords: graphical mean curvature flow
complete manifolds
non-compact Euclidean space

2010 Mathematics Subject Classification: 53C44, 53C42, 53A07

Contents

Symbols and Abbreviations

$\|s\|^2$, $\|A\|^2$, ...

Square norm of a tensor, as induced by the given metrics. For example, it is $\|s\|^2 = \sum_{i,j,k,l=1}^{m} g^{ik}g^{jl}s_{ij}s_{kl}$ and $\|A\|^2 = \sum_{i,j,k,l=1}^{m} g^{ik}g^{jl}g_{M\times N}(A_{ij}, A_{kl})$.

A

Second fundamental tensor of the graph $\Gamma(f) := \{(x, f(x)) : x \in M\}$ of f.

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A^{α}_{ij}

Components of the second fundamental tensor. At a point $p \in M$ and with respect to the tangent basis $\{e_1, \ldots, e_m\}$ and the normal basis $\{\xi_1, \ldots, \xi_n\}$, it is $A^{\alpha}_{ij} = g_{M\times N}(A(e_i, e_j), \xi_\alpha)$.

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$B(x,r)$

$B(x,r) := \{x \in \mathbb{R}^m : \|x\| < r\} \subset \mathbb{R}^m$.

δ_{ij}

Kronecker's delta, given by 1 for $i = j$ and 0 otherwise.

$\det(\psi)$

Determinant of a tensor $\psi \in \mathrm{Sym}(E^* \otimes E^*)$ with respect to a bundle metric g_E. If the bundle has rank two, it is $\det(\psi) = \frac{1}{2}\left((\mathrm{Tr}(\psi))^2 - \|\psi\|^2\right)$.

$\mathrm{D}f$, $\widetilde{\mathrm{D}}f$

The usual derivative of a function $f : \mathbb{R}^m \to \mathbb{R}^n$ in Euclidean space. The symbol D refers to the derivative with respect to a coordinate x, while $\widetilde{\mathrm{D}}$ refers to the derivative with respect to a coordinate y.

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$\mathrm{d}f$

The differential of a smooth map $f : M \to N$. Considered as a section, we have $\mathrm{d}f \in \Gamma(f^*TN \otimes T^*M)$.

$\|\mathrm{D}^k f\|^2$

For a smooth $f : \mathbb{R}^m \to \mathbb{R}^n$, it is $\|\mathrm{D}^k f\|^2 := \sum_{i_1,\dots,i_k=1}^m \langle f_{i_1,\dots,i_k}, f_{i_1,\dots,i_k}\rangle$, where the subscripts indicate the derivative with respect to the standard Euclidean basis. $\to$ p. 18

$\{e_1, \dots, e_m\}$

A local tangent frame field or a basis of T_pM, orthonormal with respect to g, respectively. $\to$ p. 15

f, F

If not stated otherwise, we use the following definition. Given two manifolds M and N, $f : M \to N$ denotes a smooth map and $F := \mathrm{id}_M \times f$ denotes the embedding of M into the product space $M \times N$.

$F(M)$

Image of M under a map F. Usually, it is $F := \mathrm{id}_M \times f$ for a smooth function $f : M \to N$, so that $F(M)$ denotes the graph of f.

$\Gamma(f)$

The graph of a function $f : M \to N$ in the product manifold $M \times N$, given by $\Gamma(f) := \{(x, f(x)) \in M \times N : x \in M\}$. $\to$ p. 25

g

Induced metric on a submanifold. If $f : M \to N$ from (M, g_M) to (N, g_N) is a smooth map and $F(x) := (x, f(x))$, then in terms terms of the metrics g_M and g_N, the induced metric is given by $\mathrm{g} = \mathrm{g}_M + f^*\mathrm{g}_N$. $\to$ pp. 9, 26, 74

$\mathrm{g}^\perp$

Restriction of $\mathrm{g}_{M\times N}$ to the normal bundle. $\to$ p. 72

$\vec{H}$

Mean curvature vector field. In local coordinates, it is given by $\vec{H} := \sum_{i,j=1}^m \mathrm{g}^{ij} A_{ij}$. $\to$ p. 10

$\lambda_1, \dots, \lambda_m$

Singular values of the differential $\mathrm{d}f$ of $f : M \to N$. $\to$ p. 15 f.

∇, ∇^{g_M}, ∇^{g_N}, $\nabla^{g_{M\times N}}$

Levi-Civita connections associated to the metrics g, g_M, g_N, $g_{M\times N}$. The first symbol, ∇, is also used for connections induced by the Levi-Civita connections. → pp. 9, 10

$\nabla^{\perp}$

Induced connection on the normal bundle. Given a smooth map $F : M \to M \times N$, it is defined as $\nabla^{\perp}_{v}\xi := \mathrm{pr}^{\perp}\left(\nabla^{g_{M\times N}}_{\mathrm{d}F(v)}\xi\right)$ for $\xi \in \Gamma(T^{\perp}M)$ and $v \in \Gamma(TM)$. → p. 10

$\star\Omega$

Jacobian of the projection π_M from the graph $\Gamma(f)$ onto the first factor of $M \times N$. → pp. 23, 43, 79

π_M, π_N

Projections from $M \times N$ onto its respective factors. → p. 25

$\mathrm{pr}^{\perp}$

Projection onto the normal bundle. If (N, g_N) is a Riemannian manifold, the submanifold is given by $F : M \to N$ and $\{e_1, \dots, e_m\}$ is a local orthonormal frame field of (M, g), it is $\mathrm{pr}^{\perp}(v) := v - \sum_{k=1}^{m} g_N(v, \mathrm{d}F(e_k))\mathrm{d}F(e_k)$ for $v \in \Gamma(F^*TN)$. → pp. 10, 71

R, R_M, R_N, $\mathrm{R}_{M\times N}$

Riemannian curvature tensors with respect to the Levi-Civita connections of the metrics g, g_M, g_N and $g_{M\times N}$. We define the curvature using the convention $\mathrm{R}(u, v)w := \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w$ and $\mathrm{R}(u_1, u_2, v_1, v_2) := g(\mathrm{R}(u_1, u_2)v_2, v_1)$. → p. 9

$\mathbb{R}^{\geq 0}$, $\mathbb{R}^{>0}$

The non-negative real numbers, $\mathbb{R}^{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$, and the positive real numbers, $\mathbb{R}^{>0} := \{x \in \mathbb{R} : x > 0\}$.

$\mathrm{Ric}\, v$, $\mathrm{Ric}(v, w)$

The Ricci operator and the Ricci tensor, as given by $\mathrm{Ric}\, v := -\sum_{k=1}^{m} \mathrm{R}(e_k, v)e_k$ for a local g-orthonormal frame $\{e_1, \dots, e_m\}$, and $\mathrm{Ric}(v, w) := g(\mathrm{Ric}\, v, w)$. → p. 28

s

Pullback of the tensor $s_{M\times N}$. In terms of the metrics g_M on M and g_N on N, it is given by $s = g_M - f^*g_N$. → p. 26

s_{ij}

Components of the tensor s. Depending on the context, this is either $\mathrm{s}_{ij} = \mathrm{s}(e_i, e_j)$ with a local frame field $\{e_1, \ldots, e_m\}$ of TM (orthonormal with respect to g), or $\mathrm{s}_{ij} = \mathrm{s}(\partial_i, \partial_j)$, where $\{\partial_1, \ldots, \partial_m\}$ are basis vectors coming from local coordinates. $\to$ p. 26

$\mathrm{s}^{\perp}$

Restriction of $\mathrm{s}_{M\times N}$ to the normal bundle $T^{\perp}M$. $\to$ p. 72

$\sec_M$, $\sec_M(v \wedge w)$

If v, w are linearly independent vectors at $p \in M$, the sectional curvature of the plane spanned by v and w is given by $\sec_M(v \wedge w) = \mathrm{R}(v, w, v, w)/\|v \wedge w\|^2$, where $\|v \wedge w\|^2 := \|v\|_{\mathrm{g}}^2 \|w\|_{\mathrm{g}}^2 - \mathrm{g}(v, w)^2$. If M is a two-dimensional surface and $\{e_1, e_2\}$ an orthonormal basis of T_pM, $\sec_M$ is the sectional curvature with respect to the plane $e_1 \wedge e_2$, i. e. $\sec_M = \sec_M(e_1 \wedge e_2)$.

$\mathrm{s}_{M\times N}$

Semi-Riemannian metric with signature (m, n) on the product $M \times N$, given by $\mathrm{s}_{M\times N} := \pi_M^* \mathrm{g}_M - \pi_N^* \mathrm{g}_N$. $\to$ p. 26

$\mathrm{Sym}(E^* \otimes E^*)$

Smooth symmetric 2-tensors on the vector bundle E.

T_N, T

In general, we use the following definition. A tensor is written upright. The subscript N indicates that it is defined on the manifold N. If $F : M \to N$ is an immersion, T is the pullback of T_N to M, i. e. it holds $\mathrm{T} := F^*\mathrm{T}_N$.

$\mathrm{Tr}(\psi)$

Trace of a tensor $\psi \in \mathrm{Sym}(E^* \otimes E^*)$ with respect to a bundle metric g_E. If E is of rank k and $\{\sigma_1, \ldots, \sigma_k\}$ is a local trivialization of E with $\mathrm{g}_E(\sigma_i, \sigma_j) = \delta_{ij}$, it holds $\mathrm{Tr}(\psi) = \sum_{i=1}^k \psi(\sigma_i, \sigma_i)$.

$\{\xi_1, \ldots, \xi_n\}$

A basis of $T_p^{\perp}M$. $\to$ p. 16

Chapter 1
Introduction

We give a rough sketch of the historic development of the topic considered in this thesis, i. e. the (graphical) mean curvature flow. There are several books and texts on the subject which provide a detailed introduction, for example [Eck04; Wan08; Man11; Smo12]. The chapter closes with an overview of the results presented in the text.

1.1 Mean Curvature Flow

In 1956, Mullins [Mul56] proposed an evolution equation to model the grain boundary of annealing metal, considering the two-dimensional case, i. e. plane curves. This evolutionary process moves every point on a curve in the direction of the mean curvature of the curve at that point. In general, given a Riemannian manifold (N, g_N) and an immersion of a closed manifold $F_0 : M \to N$, the evolution equation for the mean curvature flow is given by

$$\frac{\partial F}{\partial t}(x,t) = \vec{H}(x,t)\,, \qquad F(0,x) := F_0(x)\,, \qquad \forall x \in M\,, \quad t \in [0,T)\,, \quad (1.1.1)$$

where $\vec{H}$ is the mean curvature vector and $T > 0$ the maximal time of existence of the solution. Equation (1.1.1) is the negative gradient flow of the associated volume functional and for $N = \mathbb{R}^2$ and $\dim M = 1$ reduces to the the curve shortening flow considered by Mullins. He investigated special solutions, which included some homothetic solutions and the grim reaper, a translating solution. The self-shrinking curves were completely classified by Abresch and Langer [AL86].

In 1978, Brakke [Bra78] continued the study of equation (1.1.1) in the setting of geometric measure theory, using so-called varifolds (which are a measuretheoretic generalization of manifolds), proving existence and regularity results.

Let us shortly comment on the type of equation (1.1.1). It is a (degenerate) quasilinear parabolic equation and formally resembles a heat equation, as we will see in the following. Using the definition of the mean curvature vector and the second fundamental form, we may write

$$\vec{H} = \sum_{k,l=1}^{m} \mathrm{g}^{kl} A_{kl} = \sum_{k,l=1}^{m} \mathrm{g}^{kl} (\nabla \mathrm{d}F)(\partial_k, \partial_l)\,.$$

The right-hand side, being a trace over second derivatives, can be interpreted as a Laplace operator, which means one could write $\partial_t F = \Delta F$. Note the Laplacian depends on the time-dependent induced metric g and is therefore time-dependent itself. Also, this system is not strictly parabolic, as the principal symbol of the operator contains zeroes. This mirrors the invariance of the immersion under tangential diffeomorphisms at the level of the partial differential equation [Smo12, Section 3.1]. By a trick of DeTurck [DeT83], for compact[1] M one can equivalently consider an associated strictly parabolic system, which guarantees the short-time existence for equation (1.1.1).

The heat-equation-type nature of the mean curvature flow has the effect of regularizing the initial submanifold $F_0(M)$, and to make it "rounder". For example, Huisken proved that if $m := \dim M \geq 2$ and $F_0(M) \subset \mathbb{R}^{m+1}$ is uniformly convex (i. e. the eigenvalues of the second fundamental form are strictly positive everywhere), then $F_t(M)$ converges to a point for $t \to T$. Further, if one rescales the the $F_t(M)$ to have constant volume, the corresponding embeddings converge to a sphere for $t \to \infty$ [Hui84]. The same behavior was observed for convex curves (i. e. $\dim M = 1$) in $\mathbb{R}^2$ by Gage [Gag84] and by Gage and Hamilton [GH86].

So far, the initial data was given by a compact hypersurface, where some data on the surface was preserved under the flow. Ecker and Huisken studied entire graphs, i. e. hypersurfaces in $\mathbb{R}^{m+1}$ which are given by a height function $f : \mathbb{R}^m \to \mathbb{R}$, embedded via $F(x) := (x, f(x))$. The property of being graphic is preserved under the mean curvature flow for Lipschitz initial data with linear growth. Under an additional growth assumption of the graph at infinity, the graph asymptotically approaches a self-similar solution of the mean curvature flow. More precisely, after rescaling, the surfaces converge to a solution of the equation $F^{\perp} = \vec{H}$, which characterizes the expanding self-similar solutions of equation (1.1.1) [EH89].

In general, the mean curvature flow will develop singularities. If the initial data

[1]In this thesis, we will understand *compact* to also imply that a manifold M has empty boundary, $\partial M = \emptyset$.

is given by a smooth, uniformly convex, compact hypersurface, we have short-time existence (see e. g. [Hui84, Theorem 3.1]) and the mean curvature flow can only develop a singularity if the second fundamental form blows up for $t \to T$ [Hui84, Theorem 8.1]. By rescaling the surface near the singular point as $t \to T$ such that the second fundamental form remains uniformly bounded and using a monotonicity formula, Huisken showed that the singularities (of type I) are asymptotically self-similar [Hui90]. Therefore, finding the self-similar solutions of the mean curvature flow provides a way of classifying the possible types of singularities of type I (for planar curves, see [AL86]).

Let us now consider the case where the codimension of $F_0(M)$ is not restricted to one. In this case, the geometric situation gets more involved. A reason for this is that for $\operatorname{codim} M > 1$, the normal bundle is no longer trivial (which was the case for hypersurfaces in $\mathbb{R}^{m+1}$), but in general has a complicated structure. This can be overcome in some situations. For example, if N is a Kähler-Einstein manifold and $F : M \to N$ is a Lagrangian immersion, then the complex structure $J_N \in \operatorname{End}(TN)$ of N provides an isomorphism between $\mathrm{d}F(TM)$ and the normal bundle $T^{\perp}M$. Further, in this case the property of $F(M)$ being Lagrangian is preserved under the mean curvature flow [Smo96; Smo12].

The mean curvature flow in higher codimension can be used to obtain homotopy results for maps between manifolds. Consider a smooth map $f : M \to N$ between two Riemannian manifolds M and N. By setting

$$F : M \to M \times N\,, \qquad F(x) := (x, f(x))\,,$$

we obtain an embedding of M into the product space $M \times N$. In the case of long-time existence and convergence, the mean curvature flow of F then provides a homotopy between the initial map and the limiting map $F_\infty(M) := \lim_{t\to\infty} F_t(M)$.

M.-T. Wang showed that for a map (satisfying suitable initial conditions) between manifolds of constant curvatures $\sec_M$ and $\sec_N$ with $\sec_M \geq |\sec_N|$, the flow stays graphic and converges to a constant map [Wan02]. Also, if the initial map is a symplectomorphism and the universal covering of $M \times N$ is one of $S^2 \times S^2$, $\mathbb{R}^2 \times \mathbb{R}^2$ or $\mathbb{H}^2 \times \mathbb{H}^2$, he was able to show the long-time existence of the flow. Further, assuming that M and N have the same sectional curvature, he showed the subconvergence of the flow to a minimal Lagrangian map. This implies that any area-preserving diffeomorphism $f : M \to M$ that is homotopic to the identity can be deformed into an isometry along area preserving diffeomorphisms by the mean curvature flow [Wan01].

If the initial map is strictly area-decreasing and the manifolds are spheres, Tsui and M.-T. Wang show the convergence to a constant map. This implies that a strictly area-decreasing map is homotopic to a constant map [TW04]. The assumptions on the curvatures later were relaxed by K.-W. Lee and Y.-I. Lee [LL11] and recently by Savas-Halilaj and Smoczyk [SS14c]. Note that the conditions on the initial map in general cannot be extended to assumptions on p-volumes with $p > 2$, as noted by Guth. For example, he showed that there are maps with small 3-volume which are not homotopic to the identity [Gut13]. Let us also mention that a version of the above theorems in the pseudo-Riemannian case was obtained by Li and Salavessa [LS11].

For showing the long-time existence and convergence of the mean curvature flow, one needs to control the second fundamental form A. In codimension one, there are results by Ecker and Huisken which show the decay of A in time [EH89; EH91]. Chen, Li and Tian considered two-dimensional graphs in $\mathbb{R}^4$ and in Kähler-Einstein manifolds of the form $M_1 \times M_2$. In the flat case, assuming bounded curvature of the initial immersion and a lower bound on a projection of the induced volume form (implying a smallness condition on the first derivatives of the defining map), they showed the decay of the second fundamental form in time [CLT02]. Recently, Smoczyk, Tsui and M.-T. Wang obtained a decay estimate for the norm of the mean curvature vector under weaker assumptions on the initial map for the case where M is a flat and compact Riemann surface and N is a flat and complete Riemann surface. They also proved an estimate for the norm of the second fundamental form in the case where the initial map $f : \mathbb{T} \to \mathbb{T}$ is Lagrange [STW14].

1.2 Thesis Overview

We begin by collecting basic material and introducing notation in chapter 2. Having the necessary notions available, we use chapter 3 to state the mean curvature flow equation together with some derived evolution equations. Since we do not focus on regularity, existence and uniqueness of solutions to the flow equation, we also use this chapter to state the results which ensure good behavior in our setting. The setup of the actual geometry we consider is done in chapter 4. In particular, we introduce the main quantity which we will consider in the following two chapters. Finally, chapters 5 and 6 contain the main results, which we now describe in more detail.

Consider the mean curvature flow of two-dimensional graphs in codimension two. In general, it is helpful to know the behavior of geometric quantities under the flow. In particular, [STW14] showed that the length of the mean curvature vector is decaying as t^{-1} along the flow in the case of flat surfaces. Here, we extend this result to a more general class of non-flat Riemann surfaces.

Let $f : M \to N$ be a smooth map between Riemann surfaces. We say that the map f *evolves under the mean curvature flow*, if its graph $\Gamma(f) = \{(x, f(x)) : x \in M\} \subset M \times N$ satisfies the mean curvature flow equation (1.1.1). The map f is called *strictly area-decreasing*, if at every point $p \in M$ it is $\|\mathrm{d}f(v) \wedge \mathrm{d}f(w)\|_{\mathrm{g}_N} < \|v \wedge w\|_{\mathrm{g}_M}$ for all linearly independent $v, w \in T_pM$. Moreover, we call f *strictly length-decreasing*, if for every $p \in M$ and $v \in T_pM$ the inequality $\|\mathrm{d}f(v)\|_{\mathrm{g}_N} < \|v\|_{\mathrm{g}_M}$ holds.

To control the mean curvature vector (which is a second order quantity) in terms of a first-order quantity $\mathrm{s} := \mathrm{g}_M - f^*\mathrm{g}_N$, we first derive explicit bounds for the evolution of polynomials in the eigenvalues of s.

Theorem (Theorem 5.1.5). *Let M and N be Riemann surfaces, M being compact and N complete. Assume that there exists $\sigma \geq 0$, such that the sectional curvatures $\sec_M$ of M and $\sec_N$ of N satisfy the relation*

$$\sec_N \leq \sigma \leq \sec_M .$$

Then the following growth estimates hold for strictly area-decreasing maps along the mean curvature flow:

$$\begin{aligned}
\mathrm{Tr}(\mathrm{s}) &\geq 2\frac{\exp(\sigma t)}{\sqrt{c_1 + \exp(2\sigma t)}} && \xrightarrow[t\to\infty]{\sigma>0} 2\,, \\
\|\mathrm{s}\|^2 &\geq \frac{2}{1 + c_1\exp(-2\sigma t)} && \xrightarrow[t\to\infty]{\sigma>0} 2\,, \\
\det(\mathrm{s}) &\geq \frac{\exp(2\sigma t) - c_1}{\exp(2\sigma t) + c_1} && \xrightarrow[t\to\infty]{\sigma>0} 1\,, \\
\det(\mathrm{s}) &\leq \frac{1 + 2c_1\exp(-2\sigma t)}{1 + c_1\exp(-2\sigma t)} && \xrightarrow[t\to\infty]{\sigma>0} 1\,,
\end{aligned}$$

where $c_1 > 0$ is a constant determined by the values of $\mathrm{Tr}(\mathrm{s})$ on $M \times \{0\}$.

Note that in particular the determinant (which is not a priori positive) evolves to a purely positive quantity under the flow.

Having these bounds available, we are able to derive a decay estimate for the mean curvature vector.

Theorem (Theorem 5.2.4). *Let M and N be Riemann surfaces, M being compact and N complete. Assume that there exists $\sigma \geq 0$, such that the sectional curvatures $\sec_M$ of M and $\sec_N$ of N satisfy the relation*

$$\sec_N \leq \sigma \leq \sec_M .$$

Let $f : M \to N$ be a smooth map and evolve it by mean curvature flow. If the initial map is strictly area-decreasing and $\sigma > 0$, the mean curvature vector satisfies the estimate

$$t\|\vec{H}\|^2 \leq C$$

along the mean curvature flow for some constant $C \geq 0$, which depends on the initial values of $\operatorname{Tr}(\mathrm{s})$, *$\sigma$ and on the maximum of the sectional curvatures* $\max_{p\in M} \sec_M(p)$ *of M.*

Under the assumptions of the theorem and additionally assuming N to be compact, the preservation of the area-decreasing condition is known due to [LL11], and for $\sigma > 0$ also follows from a recent improvement in [SS14c].

We go on by deriving a decay estimate for the second fundamental form. From [STW14, Theorem 2] we know that for a Lagrangian map (with respect to a particular symplectic structure) between tori, the second fundamental form decays as t^{-1} under the mean curvature flow. We show that the same conclusion holds in a broader geometrical setting for all maps which satisfy a stronger condition on their differential.

Theorem (Theorem 5.3.12)**.** *Let (M, g_M) and (N, g_N) to be Riemann surfaces, M being compact and N complete. Assume that there exists $\sigma > 0$, such that the sectional curvatures* $\sec_M$ *of M and* $\sec_N$ *of N satisfy the relation*

$$\sec_N \leq \sigma \leq \sec_M \, .$$

Further, assume that there exist $\kappa_N, \delta_N \geq 0$, such that

$$\kappa_N := \sup_{x\in N} |\sec_N(x)| < \infty\,, \qquad \|\nabla^{\mathrm{g}_N} \mathrm{R}_N\| \leq \delta_N \, .$$

If $\operatorname{Tr}(\mathrm{s}) > \frac{2}{9}$ *is satisfied on $M \times \{0\}$, then the estimate*

$$t\|A\|^2 \leq C$$

holds along the mean curvature flow, where $C \geq 0$ is a constant depending on $\inf_{M\times\{0\}} \operatorname{Tr}(\mathrm{s})$ *and on the curvature bounds σ, δ_N, κ_N,* $\max_{p\in M} \|\nabla^{\mathrm{g}_M} \mathrm{R}_M\|(p)$ *and* $\max_{p\in M} \sec_M(p)$*.*

In chapter 6, we investigate graphs in a non-compact setting. More precisely, consider a map between two Euclidean spaces, $f_0 : \mathbb{R}^m \to \mathbb{R}^n$. If all derivatives of f_0 are bounded, i. e.

$$\sup_{x\in\mathbb{R}^m} \|\mathrm{D}^l f_0(x)\|^2 < \infty \qquad \text{for all} \quad l \geq 1\,,$$

the mean curvature flow has a short-time, smooth graphic solution f on some (maximal) time interval $[0, T)$ such that again all spatial derivatives of f are bounded for all $t \in [0, T)$ [CCH12, Proposition 5.1]. We show the following result, which may be regarded as a non-Lagrangian version of [CCH12, Theorem 1.1].

Theorem (Theorem 6.5.1). *Let $g_{\mathbb{R}^m}$ (resp. $g_{\mathbb{R}^n}$) denote the standard Euclidean metric on $\mathbb{R}^m$ (resp. $\mathbb{R}^n$). Suppose $f_0 : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz continuous function and that there exists a fixed $\delta \in (0,1]$, such that*

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^m} f_0^* g_{\mathbb{R}^n}(x) \leq (1-\delta) g_{\mathbb{R}^m} .$$

Then equation (1.1.1) *has a long-time smooth solution for all $t > 0$ with initial condition $F_0(x) := (x, f_0(x))$, such that the following statements hold.*

(i) Along the flow, the evolving submanifold stays the graph of a strictly length-decreasing map $f_t : \mathbb{R}^m \to \mathbb{R}^n$ for all $t > 0$.

(ii) The mean curvature vector of the graph satisfies the estimate

$$t \|\vec{H}\|^2 \leq C$$

for some constant $C \geq 0$.

(iii) All spatial derivatives of order $k \geq 2$ of f_t satisfy the estimate

$$t^{k-1} \sup_{x \in \mathbb{R}^m} \|\mathrm{D}^k f_t(x)\| \leq C_{k,\delta} \qquad \text{for all} \quad k \geq 2$$

and for some constants $C_{k,\delta} \geq 0$ depending on k and δ. In addition,

$$\sup_{x \in \mathbb{R}^m} \|f_t(x)\|^2 \leq \sup_{x \in \mathbb{R}^m} \|f_0(x)\|^2$$

for all $t > 0$.

If in addition f_0 satisfies $f_0(x) \to 0$ as $\|x\| \to \infty$, then $\|f_t(x)\| \to 0$ smoothly on compact sets of $\mathbb{R}^m$ for $t \to \infty$.

The thesis ends with some comments on the assumptions made in the theorems and some possible future directions. In appendix A, we collected the solutions to the differential equations used in the main text. Further, in appendix B, we give a translation of the proof of a convergence result for the solution of a particular partial differential equation. Appendix C recalls the notion of parabolic Hölder spaces, since the corresponding norms appear in the proof of theorem 6.5.1.

Chapter 2
Background Material

We recall some background material which is needed in the subsequent chapters. The material presented here is standard (see e. g. [Smo12] for a reference), and we use it mostly to set up our notation.

2.1 Submanifold Geometry

Let us recall the basics of submanifold geometry. Consider a closed manifold M of dimension $m := \dim M$, a Riemannian manifold (N, g_N) of dimension $n := \dim N$ and let $F : M \to N$ be an immersion, where we endow M with the induced metric $\mathrm{g} := F^*\mathrm{g}_N$. The *curvature endomorphism* $\mathrm{R}_N \in \Gamma(\Lambda^2 T^*N \otimes \mathrm{End}(TN))$ of a Riemannian manifold (N, g_N) is given by

$$\mathrm{R}_N(u,v)w := \nabla^{\mathrm{g}_N}_u \nabla^{\mathrm{g}_N}_v w - \nabla^{\mathrm{g}_N}_v \nabla^{\mathrm{g}_N}_u w - \nabla^{\mathrm{g}_N}_{[u,v]} w\,, \qquad u, v, w \in \Gamma(TN)\,,$$

where ∇^{g_N} is the Levi-Civita connection associated to the metric g_N. We define the *Riemannian curvature tensor* as

$$\mathrm{R}_N(u_1, u_2, v_1, v_2) := \mathrm{g}_N\left(\mathrm{R}_N(u_1, u_2)v_2, v_1\right)\,, \qquad u_1, u_2, v_1, v_2 \in \Gamma(TN)\,.$$

On (M, g) we have the curvature tensor of the Levi-Civita connection ∇ associated to g, which we denote by R. Let us define

$$\nabla^{\mathrm{g}_N}_{\mathrm{d}F(u)} \mathrm{d}F(v) := \nabla^{\mathrm{g}_N}_{\overline{\mathrm{d}F(u)}} \overline{\mathrm{d}F(v)}\,,$$

where $\overline{\mathrm{d}F(u)}$ and $\overline{\mathrm{d}F(v)}$ denote arbitrary (local) smooth extensions of the vector fields $\mathrm{d}F(u)$ and $\mathrm{d}F(v)$. Since the result of the covariant derivative does not depend on the chosen extension of the vector fields, we will identify both.

The *second fundamental tensor* of the immersion is defined as

$$A(v,w) := (\widetilde{\nabla}\mathrm{d}F)(v,w) := \nabla^{\mathrm{g}_N}_{\mathrm{d}F(v)}\mathrm{d}F(w) - \mathrm{d}F(\nabla_v w)\,,$$

where $v,w \in \Gamma(TM)$ and $\widetilde{\nabla}$ is the induced connection on $F^*TN \otimes T^*M$. Its trace with respect to the induced metric g is called the *mean curvature vector field*,

$$\vec{H} := \sum_{i=1}^{m} A(e_i, e_i)\,,$$

where $\{e_1, \ldots, e_m\}$ denotes a local orthonormal frame of TM with respect to g. The curvature tensors R and R_N are related by *Gauß' equation*, which is given by

$$\begin{aligned}\mathrm{R}(v_1,u_1,v_2,u_2) = F^*\mathrm{R}_N(v_1,u_1,v_2,u_2) &+ \mathrm{g}_N(A(v_1,v_2),A(u_1,u_2)) \\ &- \mathrm{g}_N(A(v_1,u_2),A(u_1,v_2))\,. \qquad (2.1.1)\end{aligned}$$

Defining the induced connection on the bundle $F^*TN \otimes T^*M \otimes T^*M$ to be

$$(\nabla_{\mathrm{d}F(u)}A)(v,w) := \nabla^{\mathrm{g}_{M\times N}}_{\mathrm{d}F(u)}(A(v,w)) - A(\nabla_u v, w) - A(v, \nabla_u w)\,,$$

we can write down the *Codazzi equation*,

$$\begin{aligned}(\nabla_{\mathrm{d}F(u)}A)(v,w) - (\nabla_{\mathrm{d}F(v)}A)(u,w) = \mathrm{R}_N&(\mathrm{d}F(u),\mathrm{d}F(v))\mathrm{d}F(w) \\ &- \mathrm{d}F(\mathrm{R}(u,v)w)\,. \qquad (2.1.2)\end{aligned}$$

In the remainder of this text, we will usually not use different notation for the different connections, since most of the time it is clear from the context which one to use. If it is not clear, the connections will be denoted by different symbols.

The immersion $F : M \to N$ induces an orthogonal splitting (with respect to g_N) of the pullback bundle F^*TN as

$$F^*TN = \mathrm{d}F(TM) \oplus T^{\perp}M\,.$$

On the *normal bundle* $T^{\perp}M$ we have another connection. Let us define the projection of a vector field $v \in \Gamma(F^*TN)$ onto its normal part by

$$\mathrm{pr}^{\perp} : F^*TN \to T^{\perp}M\,, \qquad \mathrm{pr}^{\perp}(v) := v - \sum_{k=1}^{m} \mathrm{g}_N(v, \mathrm{d}F(e_k))\mathrm{d}F(e_k)\,,$$

where $\{e_1, \ldots, e_m\}$ denotes a local frame for TM which is orthonormal with respect to g. We define the *normal connection* on the normal bundle as

$$\nabla^{\perp}_v \xi := \mathrm{pr}^{\perp}\left(\nabla^{\mathrm{g}_{M\times N}}_{\mathrm{d}F(v)}\xi\right)\,, \qquad \xi \in \Gamma(T^{\perp}M)\,, \quad v \in \Gamma(TM)\,.$$

With respect to the connection $\nabla^{\perp}$, the Codazzi equation (2.1.2) reads

$$(\nabla^{\perp}_u A)(v,w) - (\nabla^{\perp}_v A)(u,w) = \Big(\mathrm{R}_N(\mathrm{d}F(u),\mathrm{d}F(v))\mathrm{d}F(w)\Big)^{\perp}\,.$$

2.2 Maximum and Comparison Principles

One problem in the theory of partial differential equations is to get statements about the qualitative behavior of solutions to the equation. Since in most cases the solution to the equation cannot be obtained explicitly, we need results which describe the solution nevertheless. For the compact case, we follow the presentation in [AH11, Chapter 7.2].

Note that in local coordinates $\{x^1, \dots, x^m\}$, the Laplacian associated to the (time-dependent) metric $\mathrm{g}(t)$ is given by

$$\Delta := \sum_{i,j=1}^{m} \mathrm{g}(t)^{ij} \nabla_{\partial_i} \nabla_{\partial_j} ,$$

where in turn ∇ denotes the Levi-Civita connection associated to $\mathrm{g}(t)$. Therefore, Δ itself is time-dependent.

Proposition 2.2.1 ([AH11, Proposition 7.2]). *Suppose $\mathrm{g}(t)$, $t \in [0, T)$, is a smooth family of metric on a compact manifold M such that $u : M \times [0, T) \to \mathbb{R}$ satisfies*

$$\frac{\partial}{\partial t} u - \Delta u \geq 0 .$$

If $u \geq c$ at $t = 0$ for some $c \in \mathbb{R}$, then $u \geq c$ for all $t \geq 0$.

Proof. Let $\varepsilon > 0$ be fixed and set $u_\varepsilon := u + \varepsilon(1 + t)$. Then, by assumption, $u_\varepsilon > c$ at $t = 0$. We claim that $u_\varepsilon > c$ for all $t > 0$.

To prove this, suppose the claim is false. That is, there exists $\varepsilon > 0$ such that $u_\varepsilon \leq c$ somewhere in $M \times [0, T)$. As M is compact, there exists a first time $t_1 > 0$, such that at $(x_1, t_1) \in M \times (0, T)$ we have $u_\varepsilon(x_1, t_1) = c$ and $u_\varepsilon(x, t) \geq c$ for all $x \in M$ and $t \in [0, t_1]$. From this it follows that at (x_1, t_1) as have

$$\frac{\partial u_\varepsilon}{\partial t} \leq 0 \qquad \text{and} \qquad \Delta u_\varepsilon \geq 0 ,$$

so that

$$0 \geq \frac{\partial u_\varepsilon}{\partial t} \geq \Delta u_\varepsilon + \varepsilon > 0 ,$$

which is a contradiction. Hence, $u_\varepsilon > c$ on $M \times [0, T)$, and since $\varepsilon > 0$ is arbitrary, $u \geq c$ on $M \times [0, T)$. □

We generalize this result on a compact manifold M by considering the (semi-linear) second-order parabolic operator

$$Lu := \frac{\partial u}{\partial t} - \Delta u - \langle X(t), \nabla u \rangle - F(u, t) ,$$

where $X(t)$ is a time-dependent vector field and $F : \mathbb{R} \times [0,T) \to \mathbb{R}$ is continuous in t and locally Lipschitz in x. We say u is a *supersolution* if $Lu \geq 0$, and a *subsolution* if $Lu \leq 0$.

Theorem 2.2.2 (Comparison Principle, [AH11, Proposition 7.3]). *Suppose that u and v are differentiable with respect to time and for each $t \in [0,T)$ are in $\mathscr{C}^2(M)$. Further, suppose they satisfy $Lv \leq Lu$ on $M \times [0,T)$ and $v(x,0) \leq u(x,0)$ for all $x \in M$. Then*

$$v(x,t) \leq u(x,t)$$

holds on $M \times [0,T)$.

Proof. We apply an argument to $w := u - v$ similar to that of the previous proposition. First, compute

$$0 \leq Lu - Lv = \frac{\partial w}{\partial t} - \Delta w - \langle X, \nabla w \rangle - F(u,t) + F(v,t)\,.$$

We want to get control on the last two terms. To do this, let $\tau \in (0,t)$, so that u and v are $\mathscr{C}^2$ on $M \times [0,\tau]$. In particular, since $M \times [0,\tau]$ is compact and F is locally Lipschitz in the first argument, there exists a constant C such that

$$|F(u(x,t),t) - F(v(x,t),t)| \leq C|u(x,t) - v(x,t)| = C|w(x,t)|\,,$$

for all $(x,t) \in M \times [0,\tau]$. Now let $\varepsilon > 0$, and define $w_\varepsilon(x,t) := w(x,t) + \varepsilon \mathrm{e}^{2Ct}$. Then $w_\varepsilon(x,0) \geq \varepsilon > 0$ for all $x \in M$, while

$$\frac{\partial w_\varepsilon}{\partial t} \geq \Delta w + \langle X, \nabla w \rangle - C|w| + 2C\varepsilon \mathrm{e}^{2Ct}\,.$$

At a first point and time (x_0, t_0) where $w_\varepsilon(x_0, t_0) = 0$, we have

$$w = -\varepsilon \mathrm{e}^{2Ct_0}\,, \qquad \nabla w = 0\,, \qquad \Delta w \geq 0\,, \qquad \frac{\partial w_\varepsilon}{\partial t} \leq 0\,.$$

Therefore, at this point we conclude

$$0 \geq \frac{\partial w_\varepsilon}{\partial t} \geq \Delta w + \langle X, \nabla w \rangle - C\varepsilon \mathrm{e}^{2Ct_0} + 2C\varepsilon \mathrm{e}^{2Ct_0} \geq C\varepsilon \mathrm{e}^{2Ct_0} > 0\,,$$

which is a contradiction. Therefore, $w_\varepsilon > 0$ for all $\varepsilon > 0$, and hence $w \geq 0$ on $M \times [0,\tau]$. Since $\tau \in (0,T)$ is arbitrary, $w \geq 0$ on $M \times [0,T)$. □

From the above result we can conclude that supersolutions and subsolutions of heat type equations can be bounded by solutions to associated ordinary differential equations.

Theorem 2.2.3 (Scalar Maximum Principle, [AH11, Theorem 7.4])**.** *Suppose u is differentiable with respect to time and that for each $t \in [0, T)$, u is in $\mathscr{C}^2(M)$. Further, assume that u satisfies $Lu \geq 0$ on $M \times [0, T)$, and $u(x, 0) \geq c$ for all $x \in M$. Let $\phi(t)$ be the solution to the associated ordinary differential equation*

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = F(\phi, t), \qquad \phi(0) = c.$$

Then

$$u(x, t) \geq \phi(t)$$

for all $x \in M$ and all $t \in [0, T)$ in the interval of existence of ϕ.

Proof. Apply theorem 2.2.2 with $v(x, t) := \phi(t)$. This can be done since $Lu \geq 0 = L\phi$, and $u(x, 0) \geq c = \phi(0)$. □

Let us remark that, reversing the inequalities of the previous statements, similar results can be obtained.

We state a result which is a second derivative test for tensors and which goes back to Hamilton (see [Ham82, Theorem 9.1]). To give the statement, we first need the following definition.

Definition 2.2.4. Let (E, π, M) be a vector bundle over a manifold M with bundle metric g_E. Further, let $\psi \in \mathrm{Sym}(E^* \otimes E^*)$ be a symmetric 2-tensor on E. A real number λ is called *eigenvalue* of ψ with respect to g_E at the point $x \in M$, if there exists a non-zero vector $v \in E_x := \pi^{-1}(x)$, such that

$$\psi(v, w) = \lambda \mathrm{g}_E(v, w)$$

for any $w \in E_x$.

Lemma 2.2.5 (Second Derivative Criterion, [Ham82])**.** *Let (E, π, M) be a (possibly time-dependent) vector bundle over a compact manifold M with bundle metric g_E and metric connection ∇^E. Assume that θ is a symmetric 2-tensor on E. Further, assume there is $t_0 \in (0, T)$, such that θ is positive definite on $[0, t_0)$ and non-negative definite on $[0, t_0]$. Denote by $(x_0, t_0) \in M \times [0, T)$ the point where θ admits a null-eigenvector v. Then at (x_0, t_0) we have*

$$\theta(v, w) = 0, \quad (\nabla^E \theta)(v, v) = 0, \quad (\nabla^E_{\partial_t} \theta)(v, v) \leq 0 \quad \text{and} \quad (\Delta^E \theta)(v, v) \geq 0$$

for any $w \in E_{(x_0, t_0)} := \pi^{-1}(x_0, t_0)$.

We will also need a maximum principle for the case where M is not compact. Therefore, let M be a complete, non-compact Riemannian manifold with time dependent metric $\mathrm{g}(t)$ for $0 \leq t < T$. We denote by $B_t(p, r)$ the geodesic ball of radius r centered at p at time t.

Theorem 2.2.6 ([EH91, Theorem 4.3]). *Suppose that the manifold M with Riemannian metrics $\mathrm{g}(t)$ satisfies a uniform volume growth restriction, namely*

$$\mathrm{vol}(B_t(p,r)) \leq \exp(k(1+r^2))$$

holds for some point $p \in M$ and a uniform constant $k > 0$ for all $t \in [0,T]$.

Let f be a function on $M \times [0,T]$ which is smooth on $M \times (0,T]$ and continuous on $M \times [0,T]$. Assume that f and $\mathrm{g}(t)$ satisfy

(i) $\frac{\partial}{\partial t} f \leq \Delta f + \langle a, \nabla f \rangle + bf$ *, where the function b satisfies $\sup_{M\times[0,T]} |b| \leq \alpha_0$ for some $\alpha_0 < \infty$ and the vector field a satisfies $\sup_{M\times[0,T]} |a| \leq \alpha_1$ for some $\alpha_1 < \infty$,*

(ii) $f(p,0) \leq 0$ *for all $p \in M$,*

(iii) $\int_0^T \left(\int_M \exp(-\alpha_2^2 r_t(p,y)^2) \|\nabla f\|^2(y) \mathrm{vol}_t \right) \mathrm{d}t < \infty$ *for some $\alpha_2 > 0$,*

(iv) $\sup_{M\times[0,T]} \|\nabla_{\partial_t} \mathrm{g}\| \leq \alpha_3$ *for some $\alpha_3 < \infty$.*

Then we have $f \leq 0$ on $M \times [0,T]$.

Remark 2.2.7. We make a remark with respect to the setting which will be considered in chapter 6. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a map which satisfies $f^*\mathrm{g}_{\mathbb{R}^n} \leq \mathrm{g}_{\mathbb{R}^m}$, i. e. it is weakly length-decreasing. Further, assume that this condition is preserved when evolving f in time. We equip $\mathbb{R}^m$ with the metric $\mathrm{g} := \mathrm{g}_{\mathbb{R}^m} + f^*\mathrm{g}_{\mathbb{R}^n}$ (this is exactly the induced metric when considering the graph of f in $\mathbb{R}^m \times \mathbb{R}^n$). Then the inequality $\mathrm{g}_{\mathbb{R}^m} \leq \mathrm{g} \leq 2\mathrm{g}_{\mathbb{R}^m}$ holds, which implies

$$\begin{aligned}\mathrm{vol}(B_t(p,r)) &\leq \mathrm{vol}(B(p,2r)) \\ &\leq 2^m \mathrm{vol}(B(p,r)) = 2^m \mathrm{vol}(B(p,1)) r^m \\ &\leq \exp(k(1+r^2))\end{aligned}$$

for some $k > 0$, so that the volume growth restriction in the above theorem is satisfied. Also by the length-decreasing condition, the integral may be estimated by

$$\int_0^T \left(\int_{\mathbb{R}^m} \exp(-\alpha_2^2 r_t(p,y)^2) \|\nabla f\|^2(y) \mathrm{vol}_t \right) \mathrm{d}t \leq CT$$

for some constants $C > 0$ and $\alpha_2 > 0$, implying that (iii) holds.

2.3 Singular Value Decomposition

For a self-adjoint linear map of a vector space to itself, standard linear algebra tells that it can brought to a standard diagonal form via eigendecomposition. If one considers a map between different vector spaces, again linear algebra provides a technique, which we outline here, and which is called the *singular value decomposition theorem*.

We follow the presentation in [SS14a, Section 3.2]. Let us introduce the singular value decomposition in a way adapted to its later application. Let (M, g_M), (N, g_N) be Riemannian manifolds and consider a smooth map $f : M \to N$ between them. Fix a point $x \in M$ and let

$$\lambda_1^2(x) \le \lambda_2^2(x) \le \cdots \le \lambda_m^2(x)$$

be the eigenvalues of $f^*\mathrm{g}_N$ with respect to g_M. The corresponding values $\lambda_i \ge 0$, $i \in \{1, \ldots, m\}$, are called the *singular values* of the differential $\mathrm{d}f$ of f and give rise to continuous functions on M. Let

$$r := r(x) := \operatorname{rank} \mathrm{d}f(x)\,.$$

Obviously, $r \le \min\{m, n\}$ and $\lambda_1(x) = \cdots = \lambda_{m-r}(x) = 0$. At the point x consider an orthonormal basis $\{\alpha_1, \ldots, \alpha_{m-r}; \alpha_{m-r+1}, \ldots, \alpha_m\}$ with respect to g_M which diagonalizes $f^*\mathrm{g}_N$. Moreover, at $f(x)$ consider a basis $\{\beta_1, \ldots, \beta_{n-r};$ $\beta_{n-r+1}, \ldots, \beta_n\}$ that is orthonormal with respect to g_N, such that

$$\mathrm{d}f(\alpha_i) = \lambda_i(x)\beta_{n-m+i}\,,$$

for any $i \in \{m - r + 1, \ldots, m\}$. The above procedure is called the *singular value decomposition* of the differential $\mathrm{d}f$.

For later use, we construct a special basis for the tangent space and the normal space of the graph

$$\Gamma(f) := F(M) := \{(x, f(x)) : x \in M\}$$

in terms of the singular values. The vectors

$$\widetilde{e}_i := \begin{cases} \alpha_i\,, & 1 \le i \le m - r\,, \\ \frac{1}{\sqrt{1+\lambda_i^2(x)}}\big(\alpha_i \oplus \lambda_i(x)\beta_{n-m+i}\big)\,, & m - r + 1 \le i \le m\,, \end{cases}$$

form an orthonormal basis with respect to the metric $\mathrm{g}_{M\times N}$ of the tangent space $\mathrm{d}F(T_xM)$ of the graph $\Gamma(f)$ at x. It follows that with respect to the induced metric g, the vectors

$$e_i := \frac{1}{\sqrt{1 + \lambda_i^2(x)}}\alpha_i$$

form an orthonormal basis of T_xM. Moreover, the vectors

$$\xi_i := \begin{cases} \beta_i\,, & 1 \leq i \leq n-r\,,\\ \frac{1}{\sqrt{1+\lambda^2_{i+m-n}(x)}}(-\lambda_{i+m-n}(x)\alpha_{i+m-n} \oplus \beta_i)\,, & n-r+1 \leq i \leq n\,, \end{cases}$$

form an orthonormal basis with respect to $\mathrm{g}_{M\times N}$ of the normal space $T_x^{\perp}M$ of the graph $\Gamma(f)$ at the point x.

Chapter 3
Mean Curvature Flow

We collect some facts on mean curvature flow. Since we are not concerned with existence and uniqueness results, we state the necessary theorems which ensure good behavior in the setting considered. The material in this chapter is known and largely found in the survey article [Smo12].

3.1 Short-Time Existence, Uniqueness and Regularity

In the following chapters, we consider the case of a closed manifold M as well as the case of M being $\mathbb{R}^m$, which is non-compact. For both settings, we formulate the corresponding short-time existence results.

Let M be a closed manifold of dimension m, let $T > 0$ be a real number and denote by $F : M \times [0, T) \to (N, g_N)$ a smooth, time-dependent family of immersions of M into a Riemannian manifold (N, g_N) of dimension n, i. e. F is smooth and each

$$F_t : M \to N\,, \qquad F_t(p) := F(p,t)\,, \qquad t \in [0, T)$$

is an immersion. If F satisfies the evolution equation

$$\frac{\partial}{\partial t} F(p,t) = \vec{H}(p,t)\,, \qquad \forall p \in M\,,\ t \in [0,T)\,, \tag{3.1.1}$$

then we say that M *evolves by mean curvature flow* in N with initial data $F_0 : M \to N$.

We state the following existence and regularity result.

Theorem 3.1.1 ([Smo12, Proposition 3.2]). *Let M be a smooth closed manifold and $F_0 : M \to N$ a smooth immersion into a smooth Riemannian manifold (N, g_N). Then the mean curvature flow admits a unique smooth solution on a maximal time interval $[0, T)$, $0 < T \leq \infty$.*

For the non-compact case, we follow the discussion in [CCH12, Section 5]. If M and N are the Euclidean spaces, $M := \mathbb{R}^m$ and $N := \mathbb{R}^n$ with their respective metrics $\mathrm{g}_{\mathbb{R}^m}$ and $\mathrm{g}_{\mathbb{R}^n}$, we may consider the following non-parametric version of the mean curvature flow equation. Denote by $f_0 : \mathbb{R}^m \to \mathbb{R}^n$ a smooth map and demand

$$\frac{\partial f}{\partial t} = \sum_{i,j=1}^{m} \widetilde{\mathrm{g}}^{ij} \partial^2_{ij} f \,, \qquad f(x,0) = f_0(x) \qquad \text{for all } x \in \mathbb{R}^m \,, \tag{3.1.2}$$

where $\widetilde{\mathrm{g}}^{ij}$ denotes the components of the inverse of $\widetilde{\mathrm{g}} := \mathrm{g}_{\mathbb{R}^m} + f^* \mathrm{g}_{\mathbb{R}^n}$. If equation (3.1.2) has a smooth solution $f : \mathbb{R}^m \times [0, T) \to \mathbb{R}^n$, then equation (3.1.1) has a solution $F : \mathbb{R}^m \times [0, T) \to \mathbb{R}^m \times \mathbb{R}^n$, given by the family of graphs $(x, f(x,t))$ up to tangential diffeomorphisms (see e. g. [Bra78, Equation (4) in Chapter 3.1]).

To state the result, for a smooth $f : \mathbb{R}^m \to \mathbb{R}^n$ we define the norm

$$\|\mathrm{D}^k f\|^2 := \sum_{i_1,\ldots,i_k=1}^{m} \langle f_{i_1,\ldots,i_k}, f_{i_1,\ldots,i_k} \rangle \,,$$

where the indices denote differentiation with respect to the standard Euclidean coordinates.

Lemma 3.1.2 ([CCH12, Proposition 5.1]). *Suppose $f_0 : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth function, such that for each $l \geq 1$, we have $\sup_{x \in \mathbb{R}^m} \|\mathrm{D}^l f_0(x)\| \leq C_l$ for some constant C_l. Then* (3.1.2) *has a short-time smooth solution f on $\mathbb{R}^m \times [0, T)$ for some $T > 0$ with initial condition f_0, such that $\sup_{x \in \mathbb{R}^m} \|\mathrm{D}^l f(x,t)\| < \infty$ for every l and $t \in [0, T)$.*

Equation (3.1.2) is invariant under the following change of variables. For a fixed $\lambda > 0$, let us introduce the *parabolic scaling*

$$y := \lambda(x - x_0) \,, \qquad s := \lambda^2(t - t_0) \,.$$

For $x \in \mathbb{R}^m$, denote by $i_x : \mathbb{R}^m \to T_x\mathbb{R}^m$ the map which identifies $v \in \mathbb{R}^m$ with $\{x\} \times v \in T_x\mathbb{R}^m$ first by the inclusion $\mathbb{R}^m \hookrightarrow \{0\} \times \mathbb{R}^m$ and then by parallel translating to $\{x\} \times \mathbb{R}^m \cong T_x\mathbb{R}^m$. Then we define

$$\begin{aligned}(f_{\lambda,\kappa})(y,s) := \lambda\Big(& f(x,t) - f(x_0,t_0) \\ & - \kappa(\mathrm{D}_{i_x(x)} f)(x_0,t_0) + \kappa(\mathrm{D}_{i_{x_0}(x_0)} f)(x_0,t_0)\Big)\end{aligned}$$

$$\begin{aligned}
&= \lambda\Big(f\left(x_0 + \frac{y}{\lambda}, t_0 + \frac{s}{\lambda^2}\right) - f(x_0, t_0) \\
&\qquad - \kappa\Big(\mathrm{D}_{i_{x_0+y/\lambda}(x_0+y/\lambda)} f\Big)(x_0, t_0) \\
&\qquad + \kappa\Big(\mathrm{D}_{i_{x_0}(x_0)} f\Big)(x_0, t_0)\Big)\,.
\end{aligned}$$

We refer to $(y, f_{\lambda,\kappa}(y,s))$ as the *parabolic scaling of the graph* $(x, f(x,t))$ *by* (λ, κ) *at* (x_0, t_0). Let us denote the derivative with respect to the spatial coordinates y by $\widetilde{\mathrm{D}}$. It is $f_{\lambda,\kappa}(0,0) = 0$ and for a vector v we compute

$$(\widetilde{\mathrm{D}}_v f_{\lambda,\kappa})(y,s) = (\mathrm{D}_v f)(x,t) - \kappa(\mathrm{D}_v f)(x_0, t_0)\,,$$

which in particular implies $\widetilde{\mathrm{D}}_v f_{\lambda,0} = \mathrm{D}_v f$ and $(\widetilde{\mathrm{D}}_v f_{\lambda,1})(0,0) = 0$. For all other derivatives ($l > 1$), the relations

$$(\widetilde{\mathrm{D}}^l f_{\lambda,\kappa})(y,s) = \lambda^{1-l}(\mathrm{D}f)(x,t) \qquad \text{and} \qquad \frac{\partial f_{\lambda,\kappa}}{\partial s}(y,s) = \frac{1}{\lambda}\frac{\partial f}{\partial t}(x,t)$$

hold. Therefore, $f_{\lambda,\kappa}(y,s)$ is a solution of equation (3.1.2). Note that the second fundamental form $A_{\lambda,0}$ of the graph $(y, f_{\lambda,0}(y,s))$ is related to the second fundamental form A of the graph $(x, f(x,t))$ by

$$A_{\lambda,0}(u,v)_{|(y,s)} = \frac{1}{\lambda} A(u,v)_{|(x,t)}\,.$$

3.2 Generic Evolution Equations

For later use, we derive the evolution equations for some geometric quantities under the mean curvature flow.

Lemma 3.2.1. *Let M be a manifold of dimension m, let (N, g_N) be a Riemannian manifold of dimension $n \geq m$ and let $F : M \to N$ denote a smooth solution to the mean curvature flow. Let $\mathrm{h}_N \in \mathrm{Sym}(T^*N \otimes T^*N)$ be a symmetric tensor on N and denote by $\mathrm{h} := F_t^* \mathrm{h}_N$ its pullback. Then the time derivative of the trace of h is given by*

$$\frac{\partial}{\partial t}\,\mathrm{Tr}(\mathrm{h}) = 2\sum_{k,l=1}^{m} \mathrm{g}_N\left(\vec{H}, A(e_k, e_l)\right)\mathrm{h}(e_k, e_l) + 2\sum_{k=1}^{m} \mathrm{h}\left(\nabla_{\mathrm{d}F(e_k)}\vec{H}, \mathrm{d}F(e_k)\right)\,,$$

where $\{e_1, \ldots, e_m\}$ denotes a local frame which is orthonormal with respect to the induced metric $\mathrm{g} = F_t^* \mathrm{g}_N$.

Proof. Let us choose local coordinates $\{x^1, \ldots, x^m\}$, such that the components of the metric and the second fundamental form are given by

$$\mathrm{g}_{ij} := \mathrm{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \qquad \text{and} \qquad A_{kl} := A\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)\,,$$

and further denote the components of the inverse metric by g^{ij}. Then, the evolution of the components of the inverse metric is given by

$$\nabla_{\partial_t}\mathrm{g}^{ij} = -2\sum_{k,l=1}^{m} \mathrm{g}^{ik}\mathrm{g}^{jl}\mathrm{g}_N\left(\vec{H}, A_{kl}\right) .$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial t}\operatorname{Tr}(\mathrm{h}) &= \frac{\partial}{\partial t}\sum_{i,j=1}^{m}\mathrm{g}^{ij}\mathrm{h}_{ij}\\ &= 2\sum_{k,l=1}^{m}\sum_{i,j=1}^{m}\mathrm{g}^{ik}\mathrm{g}^{jl}\mathrm{g}_N\left(\vec{H}, A_{kl}\right)\mathrm{h}_{ij}\\ &\qquad\qquad + 2\sum_{i,j=1}^{m}\mathrm{g}^{ij}\mathrm{h}_N\left(\nabla_{\mathrm{d}F(\partial_i)}\vec{H}, \mathrm{d}F\left(\frac{\partial}{\partial x^j}\right)\right) .\end{aligned}$$

Choosing a basis which is orthonormal with respect to g, we obtain $\mathrm{g}^{ij} = \delta^{ij}$, which shows the claim. □

We will also need the following equations for the second-order quantities $\vec{H}$ and A. Let $\{e_1, \ldots, e_m\}$ denote a local g-orthonormal frame of TM, and, choosing a local trivialization $\{\xi_1, \ldots, \xi_n\}$ of the normal bundle around a point (x_0, t_0), also define $A^{\alpha}_{ij} := \mathrm{g}_{M\times N}(A(e_i, e_j), \xi_\alpha)$.

Lemma 3.2.2 ([Smo12, Corollary 3.8]). *Let M be a manifold of dimension m, let (N, g_N) be a Riemannian manifold of dimension $n \geq m$ and let $F : M \to N$ denote a smooth solution to the mean curvature flow. The mean curvature vector satisfies the following evolution equations:*

$$\begin{aligned}(\nabla_{\partial_t} - \Delta)\vec{H} &= -\sum_{i,j,k=1}^{m}\mathrm{g}^{ij}(\partial_t\Gamma^k_{ij})\mathrm{d}F(e_k)\\ &\quad + 2\sum_{i,j=1}^{m}\mathrm{g}_N\Big(A(e_i, e_j), \vec{H}\Big)A(e_i, e_j)\\ &\quad + \sum_{i=1}^{m}\mathrm{R}_N(\vec{H}, \mathrm{d}F(e_i))\mathrm{d}F(e_i) ,\\ \left(\frac{\partial}{\partial t} - \Delta\right)\|\vec{H}\|^2 &= -2\|\nabla^{\perp}\vec{H}\|^2 + 2\sum_{i,j=1}^{m}\Big(\mathrm{g}_N(A(e_i, e_j), \vec{H})\Big)^2\\ &\qquad + 2\sum_{k=1}^{m}\mathrm{R}_N(\vec{H}, \mathrm{d}F(e_k), \vec{H}, \mathrm{d}F(e_k)) .\end{aligned}$$

Corollary 3.2.3. *Let (M, g_M) and (N, g_N) be manifolds of dimensions $m := \dim M \leq \dim N$ and consider a smooth solution $F : M \to N$ of the mean curvature flow. Then the estimate*

$$\left(\frac{\partial}{\partial t} - \Delta\right) \|\vec{H}\|^2 \leq -2\|\nabla^\perp \vec{H}\|^2 + 2\|\vec{H}\|^2\|A\|^2 \\ + 2\sum_{k=1}^{m} \mathrm{R}_N(\vec{H}, \mathrm{d}F(e_k), \vec{H}, \mathrm{d}F(e_k)) \tag{3.2.1}$$

holds whenever the flow is defined.

Proof. Using

$$\sum_{i,j=1}^{m} \Big(g_N(A(e_i, e_j), \vec{H})\Big)^2 \leq \|\vec{H}\|^2\|A\|^2 ,$$

the claim follows immediately from lemma 3.2.2. □

Lemma 3.2.4 ([Smo12, Corollary 3.9])**.** *Under the mean curvature flow the quantity $\|A\|^2$ satisfies the following evolution equation:*

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) \|A\|^2 &= -2\|\nabla^\perp A\|^2 \\
&+ 2\sum_{i,j,k,l=1}^{m} \Big(g_N(A_{ij}, A_{kl})\Big)^2 + 2\sum_{i,j,k=1}^{m}\sum_{\alpha,\beta=1}^{n} \Big(A^\alpha_{ij}A^\beta_{jk} - A^\beta_{ik}A^\alpha_{jk}\Big)^2 \\
&+ 4\sum_{i,j,k,l=1}^{m} (F^*\mathrm{R}_N)_{kilj}\Big(g_N(A_{ij}, A_{kl}) - \delta^{kl}\sum_{p=1}^{m} g_N(A_{ip}, A_{jp})\Big) \\
&+ 8\sum_{i,k,l=1}^{m} \mathrm{R}_N(A_{kl}, A_{ik}, \mathrm{d}F(e_l), \mathrm{d}F(e_i)) \\
&+ 2\sum_{i,k,l=1}^{m} \mathrm{R}_N(A_{kl}, \mathrm{d}F(e_i), A_{kl}, \mathrm{d}F(e_i)) \\
&+ 2\sum_{i,l,k=1}^{m} (\nabla_{\mathrm{d}F(e_i)}\mathrm{R}_N)(A_{kl}, \mathrm{d}F(e_l), \mathrm{d}F(e_k), \mathrm{d}F(e_i)) \\
&+ 2\sum_{i,l,k=1}^{m} (\nabla_{\mathrm{d}F(e_k)}\mathrm{R}_N)(A_{kl}, \mathrm{d}F(e_i), \mathrm{d}F(e_l), \mathrm{d}F(e_i)) .
\end{aligned}$$

3.3 A Note on Singularities

The mean curvature flow in general produces singularities, which already may be seen by the standard example of a shrinking sphere. Let $\iota : S^n(R) \to \mathbb{R}^{n+1}$ be the inclusion of the sphere with radius $R > 0$ into Euclidean space, centered at the origin. The mean curvature flow with this initial data,

$$\frac{\partial}{\partial t} F_t(x,t) = \vec{H}(x,t), \qquad F_0(S^n(R)) = \imath(S^n(R)),$$

then reduces to the ordinary differential equation

$$\frac{\partial}{\partial t} r = -\frac{n}{r},$$

where r is the radius of the sphere, and with solution given by

$$r(t) = \sqrt{R^2 - 2nt}.$$

Note that the solution only exists for $t \in [0, R^2/2n)$. While the sphere is shrinking under the flow, the second fundamental form is given by

$$\|A\|^2(t) = \frac{n}{(r(t))^2} = \frac{n}{R^2 - 2nt} = \frac{1/2}{R^2/2n - t},$$

so that it blows up as $t \to R^2/2n$.

If the domain M is compact, the second fundamental form can be used to detect singularities [Hui90; Smo12]. A singularity occurs if the mean curvature flow only exists for finite time, which means by [Smo12, Proposition 3.11] that

$$\limsup_{t \to T} \max_{F_t(M)} \|A\|^2 = \infty.$$

It is possible to classify singularities by the blow-up rate of $\max_{F_t(M)} \|A\|^2$ as follows. One says that $F_0(M)$ develops a singularity of *type I*, if there exists a constant $C > 0$, such that

$$\sup_{F_t(M)} \|A\|^2 \leq \frac{C}{T-t}, \qquad \forall t \in [0, T).$$

Otherwise one calls the singularity of *type II*. Using this definition, the sphere in the above example develops a type I singularity.

In the setting considered in chapter 5, $\|A\|^2$ stays bounded, so that under the mean curvature flow no singularity formation occurs. To give the precise statement, we need to introduce the following notions.

Let (M, g_M) and (N, g_N) be two Riemannian manifolds, where M is compact and N complete. Consider a smooth map $f : M \to N$. The embedding into

$M \times N$ is given by $F : M \to M \times N$, $F(x) := (x, f(x))$. The *associated Jacobian* of the projection from $\pi_M : F(M) \to M$ is defined via

$$\pi_M^* \Omega_M = \mathrm{Jac}(\pi_M) \Omega_{F(M)} ,$$

where Ω_M is the volume form on (M, g_M) and $\Omega_{F(M)}$ is the volume form on M induced by g. Setting $\Omega := (\pi_M \circ F)^* \Omega_M$ and using the Hodge star of the induced metric g, we may write

$$\mathrm{Jac}(\pi_M) = \star \Omega .$$

Theorem 3.3.1 ([Wan02]). *Let (M, g_M) and (N, g_N) be compact Riemannian manifolds. Further, let $f : M \to N$ be a smooth map and let $F := \mathrm{id}_M \times f : M \to M \times N$ evolve by the mean curvature flow. Assume that the differential inequality*

$$\left(\frac{\partial}{\partial t} - \Delta \right) \star \Omega \geq \delta \|A\|^2$$

holds for some $\delta > 0$ and all $t \in [0, T)$. Then the mean curvature flow has long-time existence.

To connect this result to the setting examined in chapter 5, let us cite the discussion given in [SS14b].

The proof of the above theorem uses White's regularity theorem [Whi05] to exclude finite time singularities. This regularity theorem can also be applied if on finite time intervals the graph stays in compact regions of $M \times N$. Assume that the length of the mean curvature vector is bounded, i. e.

$$\|\vec{H}\| \leq C$$

for some constant $C \geq 0$ and for all $t \in [0, T)$. Further, fix a point $x \in M$ and for $t_0, t_1 \in [0, T)$ with $t_0 \leq t_1$, consider the smooth curve $\gamma : [t_0, t_1] \to M \times N$, given by

$$\gamma(x, t) := F(x, t) .$$

The length $L(\gamma)$ of γ can be estimated using the bound on the mean curvature vector,

$$L(\gamma) := \int_{t_0}^{t_1} \left\| \frac{\partial F}{\partial t}(x, t) \right\| \mathrm{d}t \leq \int_{t_0}^{t_1} \|\vec{H}(x, t)\| \mathrm{d}t \leq C(t_1 - t_0) \leq CT .$$

Therefore,

$$\mathrm{dist}\Big(F(x, t_0), F(x, t_1) \Big) \leq L(\gamma) \leq CT ,$$

which implies that on finite time intervals, the graph stays in compact regions W of $M \times N$.

By Nash's embedding theorem [Nas56], one can embed W isometrically in some euclidean space $\mathbb{R}^k$ and make sure that the embedding has bounded geometry. Therefore, White's regularity theorem for the mean curvature flow with controlled error terms can be applied to the mean curvature flow of $F(M) \subset W \subset \mathbb{R}^k$. Then by the same arguments as developed in [Wan02, Section 4], one can prove the long-time existence of the mean curvature flow.

Chapter 4
Graphical Mean Curvature Flow

We use this short chapter to introduce the graphical setting for the mean curvature flow. Further, we give the definition of a special tensor (first introduced in [Smo04]) which will be analyzed in the subsequent chapters. Therefore, we also derive evolution equations for this tensor and related quantities.

4.1 Geometry of Graphs

Let us specialize the generic statements for submanifolds found in section 2.1 to the graphic case. Assume (M, g_M) and (N, g_N) to be Riemannian manifolds of dimensions m and n, respectively. The induced metric on the product manifold will be denoted by

$$\mathrm{g}_{M\times N} := \mathrm{g}_M \times \mathrm{g}_N \,.$$

A smooth map $f : M \to N$ defines an embedding $F : M \to M \times N$, given by

$$F(x) := (x, f(x))\,, \qquad x \in M\,.$$

The *graph of f* is defined to be the submanifold

$$\Gamma(f) := F(M) := \{(x, f(x)) : x \in M\} \qquad \subset M \times N\,.$$

Since F is an embedding, it induces another Riemannian metric $\mathrm{g} := F^*\mathrm{g}_{M\times N}$ on M. The two natural projections

$$\pi_M : M \times N \to M\,, \qquad \pi_N : M \times N \to N$$

are submersions, that is they are smooth and have maximal rank. Note that the tangent bundle of the product manifold $M \times N$ splits as a direct sum

$$T(M \times N) = TM \oplus TN .$$

The four metrics g_M, g_N, $\mathrm{g}_{M\times N}$ and g are related by

$$\begin{aligned} \mathrm{g}_{M\times N} &= \pi_M^* \mathrm{g}_M + \pi_N^* \mathrm{g}_N , \\ \mathrm{g} &= F^* \mathrm{g}_{M\times N} = \mathrm{g}_M + f^* \mathrm{g}_N . \end{aligned}$$

Let us further define the symmetric 2-tensors

$$\begin{aligned} \mathrm{s}_{M\times N} &:= \pi_M^* \mathrm{g}_M - \pi_N^* \mathrm{g}_N , \\ \mathrm{s} &:= F^* \mathrm{s}_{M\times N} = \mathrm{g}_M - f^* \mathrm{g}_N . \end{aligned}$$

Note that $\mathrm{s}_{M\times N}$ is a semi-Riemannian metric of signature (m, n) on the product manifold $M \times N$.

From the construction of the basis in section 2.3, we deduce that at a fixed point $x \in M$ it is

$$\mathrm{s}(e_i, e_j) = \mathrm{s}_{M\times N}(\widetilde{e}_i, \widetilde{e}_j) = \frac{1-\lambda_i^2(x)}{1+\lambda_i^2(x)}\delta_{ij} , \qquad 1 \le i, j \le m ,$$

with δ_{ij} given by

$$\delta_{ij} := \begin{cases} 1 , & i = j , \\ 0 , & i \neq j . \end{cases}$$

Therefore, the eigenvalues of the 2-tensor s with respect to g are given by

$$\frac{1-\lambda_1^2(x)}{1+\lambda_1^2(x)} \geq \cdots \geq \frac{1-\lambda_{m-1}^2(x)}{1+\lambda_{m-1}^2(x)} \geq \frac{1-\lambda_m^2(x)}{1+\lambda_m^2(x)} . \tag{4.1.1}$$

Moreover, with $r := \operatorname{rank}(\mathrm{d}f)$ we have

$$\mathrm{s}_{M\times N}(\xi_i, \xi_j) = \begin{cases} -\delta_{ij} , & 1 \le i \le n - r , \\ -\frac{1-\lambda_{i+m-n}^2(x)}{1+\lambda_{i+m-n}(x)}\delta_{ij} , & n - r + 1 \le i \le n , \end{cases} \tag{4.1.2}$$

and

$$\mathrm{s}_{M\times N}(\widetilde{e}_i, \xi_j) = -\frac{2\lambda_{m-r+i}(x)}{1+\lambda_{m-r+i}^2(x)}\delta_{ij} , \qquad 1 \le i, j \le r .$$

The Levi-Civita connection $\nabla^{g_{M\times N}}$ associated to the Riemannian metric $g_{M\times N}$ on $M \times N$ is related to the Levi-Civita connections ∇^{g_M} on (M, g_M) and ∇^{g_N} on (N, g_N) by

$$\nabla^{g_{M\times N}} = \pi_M^* \nabla^{g_M} \oplus \pi_N^* \nabla^{g_N} .$$

The corresponding curvature operator $\mathrm{R}_{M\times N}$ on $M \times N$ with respect to the metric $g_{M\times N}$ is related to the curvature operators R_M on (M, g_M) and R_N on (N, g_N) by

$$\mathrm{R}_{M\times N} = \pi_M^* \mathrm{R}_M \oplus \pi_N^* \mathrm{R}_N .$$

We denote by ∇ the Levi-Civita connection on M with respect to the induced metric g and the corresponding curvature tensor by R.

The *second fundamental tensor* A of the graph $\Gamma(f)$ is defined as

$$A(v, w) := (\widetilde{\nabla} \mathrm{d}F)(v, w) := \nabla^{g_{M\times N}}_{\mathrm{d}F(v)} \mathrm{d}F(w) - \mathrm{d}F(\nabla_v w) ,$$

where $v, w \in \Gamma(TM)$ and $\widetilde{\nabla}$ is the induced connection on $F^*T(M \times N) \otimes T^*M$. The trace of A with respect to the metric g is called the *mean curvature vector field* of $\Gamma(f)$ and it will be denoted by

$$\vec{H} := \operatorname{Tr} A := \sum_{i=1}^{m} A(e_i, e_i) ,$$

where $\{e_1, \ldots, e_m\}$ is an arbitrary orthonormal frame of TM. Note that $\vec{H}$ is a section in the normal bundle $T^{\perp}M$ of the graph.

Definition 4.1.1. Let (M, g_M) and (N, g_N) be Riemannian manifolds. A smooth map $f : M \to N$ is called *weakly k-volume-decreasing* if

$$\|(\Lambda^k \mathrm{d}f)(v_1, \ldots, v_k)\|_{g_N} := \|\mathrm{d}f(v_1) \wedge \cdots \wedge \mathrm{d}f(v_k)\|_{g_N} \leq 1 ,$$

for all $\{v_1, \ldots, v_k\} \in \Gamma(TM)$ orthonormal with respect to g_M. If the inequality is strict, i. e. $\|(\Lambda^k \mathrm{d}f)(v_1, \ldots, v_k)\| < 1$, then f is called *strictly k-volume-decreasing*. As usual, for $k = 1$ we use the term *length* instead of 1-volume and if $k = 2$ we use the term *area* instead of 2-volume. The map f is called an *isometric immersion*, if $f^* g_N = g_M$.

In terms of the singular values, the condition of a map being weakly length-decreasing may be expressed as

$$\lambda_i^2(x) \leq 1$$

for $i = 1, \dots, m$ and $x \in M$ (or $\lambda_i^2(x) < 1$). Since the eigenvalues of the tensor s are given by

$$\frac{1-\lambda_i^2(x)}{1+\lambda_i^2(x)}\,,$$

the (strict) length-decreasing property of the map f is equivalent to the (strict) positivity of the symmetric tensor s.

In the area-decreasing case, the conditions on the eigenvalues are given by

$$\lambda_i^2(x)\lambda_j^2(x) \le 1 \qquad \text{and} \qquad \lambda_i^2(x)\lambda_j^2(x) < 1\,,$$

for any $1 \le i < j \le m$ and $x \in M$, respectively. In terms of the eigenvalues of the tensor s, note that the sum of two of these is given by

$$\frac{1-\lambda_i^2(x)}{1+\lambda_i^2(x)} + \frac{1-\lambda_j^2(x)}{1+\lambda_j^2(x)} = \frac{2(1-\lambda_i^2(x)\lambda_j^2(x))}{(1+\lambda_i^2(x))(1+\lambda_j^2(x))}\,,$$

so that the (strictly) area-increasing property of the map f corresponds to the (strict) 2-positivity of the tensor s.

Let us define the tensor

$$\Phi_c := \mathrm{s} - \frac{1-c}{1+c}\mathrm{g}\,.$$

This tensor satisfies the following differential equation.

Lemma 4.1.2 ([SS14a, Lemma 3.2]). *The tensor Φ_c satisfies the equation*

$$\begin{aligned}(\Delta\Phi_c)(v,w) &= \mathrm{s}_{M\times N}\left(\nabla_{\mathrm{d}F(v)}\vec{H}, \mathrm{d}F(w)\right) + \mathrm{s}_{M\times N}\left(\nabla_{\mathrm{d}F(w)}\vec{H}, \mathrm{d}F(v)\right) \\ &\quad + 2\frac{1-c}{1+c}\mathrm{g}_{M\times N}\left(\vec{H}, A(v,w)\right) \\ &\quad + \Phi_c\left(\operatorname{Ric} v, w\right) + \Phi_c\left(\operatorname{Ric} w, v\right) \\ &\quad + 2\sum_{k=1}^m \left(\mathrm{s}_{M\times N} - \frac{1-c}{1+c}\mathrm{g}_{M\times N}\right)\left(A(e_k,v), A(e_k,w)\right) \\ &\quad + \frac{4}{1+c}\sum_{k=1}^m \left(f^*\mathrm{R}_N(e_k,v,e_k,w) - c\mathrm{R}_M(e_k,v,e_k,w)\right),\end{aligned}$$

where

$$\operatorname{Ric} v := -\sum_{k=1}^m \mathrm{R}(e_k,v)e_k$$

is the Ricci operator on (M, g) and $\{e_1, \dots, e_m\}$ is any orthonormal frame with respect to the induced metric g.

4.2 Evolution Equations

We derive the evolution equation for the tensor s and its trace. For the tensor s itself, the following equation holds.

Lemma 4.2.1. *Under the mean curvature flow, the tensor* s *obeys the equation*

$$
\begin{aligned}
\Big((\nabla_{\partial_t} - \Delta)\,\mathrm{s}\Big)(v,w) = &- \mathrm{s}(\mathrm{Ric}\,v,w) - \mathrm{s}(\mathrm{Ric}\,w,v)\\
&- 2\sum_{k=1}^{m} \mathrm{s}_{M\times N}\Big(A(e_k,v),A(e_k,w)\Big)\\
&- 2\sum_{k=1}^{m} (f^*\mathrm{R}_N - \mathrm{R}_M)(e_k,v,e_k,w)\,.
\end{aligned}
$$

Proof. From lemma 4.1.2 (setting $c := 1$) we see that the Laplacian of s is given by

$$
\begin{aligned}
(\Delta \mathrm{s})\,(v,w) = {} & \mathrm{s}_{M\times N}\left(\nabla_{\mathrm{d}F(v)}\vec{H},\mathrm{d}F(w)\right) + \mathrm{s}_{M\times N}\left(\nabla_{\mathrm{d}F(w)}\vec{H},\mathrm{d}F(v)\right)\\
&+ \mathrm{s}\,(\mathrm{Ric}\,v,w) + \mathrm{s}\,(\mathrm{Ric}\,w,v)\\
&+ 2\sum_{k=1}^{m} \mathrm{s}_{M\times N}\,(A(e_k,v),A(e_k,w))\\
&+ 2\sum_{k=1}^{m}\Big(f^*\mathrm{R}_N(e_k,v,e_k,w) - \mathrm{R}_M(e_k,v,e_k,w)\Big)\,.
\end{aligned}
$$

Using that the second fundamental form (with respect to the manifold $M\times[0,T)$) is symmetric, the time-derivative of s is

$$
(\nabla_{\partial_t}\mathrm{s})(v,w) = \mathrm{s}_{M\times N}\Big(\nabla_{\mathrm{d}F(v)}\vec{H},\mathrm{d}F(w)\Big) + \mathrm{s}_{M\times N}\Big(\mathrm{d}F(v),\nabla_{\mathrm{d}F(w)}\vec{H}\Big)\,.
$$

The claim follows from combining these two formulas. □

Lemma 4.2.2. *Under the mean curvature flow, the trace of the tensor* s *obeys the equation*

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)\mathrm{Tr}(\mathrm{s})\\
&= -2\sum_{k,l=1}^{m}\left(\mathrm{s}_{M\times N} - \frac{1-\lambda_k^2}{1+\lambda_k^2}\mathrm{g}_{M\times N}\right)\Big(A(e_k,e_l),A(e_k,e_l)\Big)\\
&\quad + 2\sum_{k,l=1}^{m}\left(\frac{2\lambda_k^2}{1+\lambda_k^2}\mathrm{R}_M(e_k,e_l,e_k,e_l) - \frac{2}{1+\lambda_k^2}f^*\mathrm{R}_N(e_k,e_l,e_k,e_l)\right),
\end{aligned}
\tag{4.2.1}
$$

where $\{e_1,\ldots,e_m\}$ *denotes the orthonormal frame field with respect to* g *that is constructed in section 2.3.*

Proof. From the Gauß equation (2.1.1) we obtain

$$
\begin{aligned}
\mathrm{s}(\operatorname{Ric} e_k, e_k) = \frac{1-\lambda_k^2}{1+\lambda_k^2} \Bigg\{ & \sum_{l=1}^{m} \Big(\mathrm{R}_M(e_k, e_l, e_k, e_l) + f^* \mathrm{R}_N(e_k, e_l, e_k, e_l) \Big) \\
& - \sum_{l=1}^{m} \mathrm{g}_{M\times N} \Big(A(e_k, e_l), A(e_k, e_l) \Big) \\
& + \mathrm{g}_{M\times N} \Big(\vec{H}, A(e_k, e_k) \Big) \Bigg\} ,
\end{aligned}
$$

so that

$$
\begin{aligned}
(\Delta \mathrm{s})(e_k, e_k) &= 2 \mathrm{s}_{M\times N} \Big(\nabla_{\mathrm{d}F(e_k)} \vec{H}, \mathrm{d}F(e_k) \Big) \\
&+ 2 \frac{1-\lambda_k^2}{1+\lambda_k^2} \mathrm{g}_{M\times N} \Big(\vec{H}, A(e_k, e_k) \Big) \\
&+ 2 \sum_{l=1}^{m} \left(\mathrm{s}_{M\times N} - \frac{1-\lambda_k^2}{1+\lambda_k^2} \mathrm{g}_{M\times N} \right) \Big(A(e_k, e_l), A(e_k, e_l) \Big) \\
&- 2 \sum_{l=1}^{m} \left(\frac{2\lambda_k^2}{1+\lambda_k^2} \mathrm{R}_M(e_k, e_l, e_k, e_l) - \frac{2}{1+\lambda_k^2} f^* \mathrm{R}_N(e_k, e_l, e_k, e_l) \right) .
\end{aligned}
$$

Summing over k and using lemma 3.2.1, the claim follows. □

Chapter 5
Curvature Decay Estimates

We consider the mean curvature flow of maps between Riemann surfaces. From chapter 3 we know that for a compact immersion M the flow has short-time existence. Therefore we may consider the behavior of various quantities associated to the immersion. We prove that under certain curvature assumptions and if the initial map is strictly area-decreasing, the tensor s satisfies certain growth estimates. Then we use these to derive decay estimates for the mean curvature vector and the second fundamental tensor.

5.1 Two-Dimensional Graphs and Estimates for the Singular Values

In this chapter, we assume (M, g_M) and (N, g_N) to be Riemannian manifolds with $\dim M = \dim N = 2$, i. e. M and N are *Riemann surfaces*. Let us recall the values of the tensor s from section 4.1 when evaluated at a fixed point $x \in M$ with respect to the bases $\{e_1, e_2\}$ and $\{\xi_1, \xi_2\}$ defined in section 2.3. First, note that $\{e_1, e_2\}$ diagonalizes the tensor s with eigenvalues

$$\mathrm{s}(e_1, e_1) = \frac{1 - \lambda_1^2}{1 + \lambda_1^2} \geq \frac{1 - \lambda_2^2}{1 + \lambda_2^2} = \mathrm{s}(e_2, e_2)\,.$$

Further, with respect to the normal basis $\{\xi_1, \xi_2\}$, the tensor $\mathrm{s}_{M \times N}$ on $M \times N$ satisfies the relations

$$\mathrm{s}_{M\times N}(\xi_i, \xi_j) = -\frac{1 - \lambda_i^2}{1 + \lambda_i^2}\delta_{ij}$$

and

$$\mathrm{s}_{M\times N}(\mathrm{d}F(e_i),\xi_j) = \mathrm{s}_{M\times N}(\xi_i,\mathrm{d}F(e_j)) = -\frac{2\lambda_i}{1+\lambda_i^2}\delta_{ij}\,.$$

Let us also comment on the local computations carried out in this chapter. Fix a point $(x_0,t_0)\in M\times[0,T)$. In any of the local calculations, consider a coordinate system $(x_1,x_2;t)$ around (x_0,t_0) such that

$$\frac{\partial}{\partial x^k}(x_0) = e_k(x_0,t_0) \qquad \text{for} \quad k=1,2\,,$$

where $\{e_1,e_2\}$ denotes the orthonormal frame constructed in section 2.3. Then all formulas are valid at (x_0,t_0) with this choice of coordinates.

Now assume that there exists a constant $\sigma>0$, such that the curvatures of (M,g_M) and (N,g_N) are subject to the conditions

$$\sec_M > -\sigma\,, \qquad \mathrm{Ric}_M \geq (m-1)\sigma \geq (m-1)\sec_N\ .$$

In this situation (for compact M and N), we know by [SS14c, Theorem A], that the mean curvature flow of a graph of a strictly area-decreasing map stays the graph of a strictly area-decreasing map. If $\dim M=\dim N=2$, the strictly area-decreasing condition is equivalent to

$$0 < \frac{2(1-\lambda_1^2\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)} = \mathrm{Tr}(\mathrm{s})\,.$$

In the following, we will consider the logarithm of the trace of s.

The following lemmas and their corollaries carry out calculations which are analogous to [STW14, Proposition 3.3], but take all curvature terms into consideration.

Lemma 5.1.1. *Let (M,g_M) and (N,g_N) be two Riemann surfaces. Assume that the map $f: M\to N$ is strictly area-decreasing and evolve it by mean curvature flow. Then, as long as the flow exists, the trace* $\mathrm{Tr}(\mathrm{s})$ *of* s *satisfies the equation*

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)\ln\mathrm{Tr}(\mathrm{s}) \\
&\quad = 2\|A\|^2+\frac{1}{2}\|\nabla\ln\mathrm{Tr}(\mathrm{s})\|^2 \\
&\qquad + \frac{2}{(\mathrm{Tr}(\mathrm{s}))^2}\sum_{i=1}^{2}\left\{\frac{2\lambda_2}{1+\lambda_2^2}\mathrm{g}_{M\times N}\Big(A(e_1,e_i),\xi_1\Big)\right. \\
&\qquad\qquad\qquad\qquad \left.+\frac{2\lambda_1}{1+\lambda_1^2}\mathrm{g}_{M\times N}\Big(A(e_2,e_i),\xi_2\Big)\right\}^2 \\
&\qquad + \frac{2}{\mathrm{Tr}(\mathrm{s})}\sum_{k,l=1}^{2}\left\{\frac{2\lambda_k^2}{1+\lambda_k^2}\mathrm{R}_M(e_k,e_l,e_k,e_l)-\frac{2}{1+\lambda_k^2}f^*\mathrm{R}_N(e_k,e_l,e_k,e_l)\right\}.
\end{aligned} \tag{5.1.1}$$

Proof. First, the evolution for the logarithm of the trace is given by

$$\left(\frac{\partial}{\partial t}-\Delta\right)\ln\mathrm{Tr}(\mathrm{s})=\frac{1}{\mathrm{Tr}(\mathrm{s})}\left(\frac{\partial}{\partial t}-\Delta\right)\mathrm{Tr}(\mathrm{s})+\|\nabla\ln\mathrm{Tr}(\mathrm{s})\|^2.$$

The first term is essentially given by lemma 4.2.2, so that

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)\ln\mathrm{Tr}(\mathrm{s}) \\
&\quad = -\frac{2}{\mathrm{Tr}(\mathrm{s})}\sum_{k,l=1}^{2}\left(\mathrm{s}_{M\times N}-\frac{1-\lambda_k^2}{1+\lambda_k^2}\mathrm{g}_{M\times N}\right)\Big(A(e_k,e_l),A(e_k,e_l)\Big) \\
&\qquad + \frac{2}{\mathrm{Tr}(\mathrm{s})}\sum_{k,l=1}^{2}\left\{\frac{2\lambda_k^2}{1+\lambda_k^2}\mathrm{R}_M(e_k,e_l,e_k,e_l)\right. \\
&\qquad\qquad\qquad\qquad \left.-\frac{2}{1+\lambda_k^2}f^*\mathrm{R}_N(e_k,e_l,e_k,e_l)\right\} \\
&\qquad + \|\nabla\ln\mathrm{Tr}(\mathrm{s})\|^2.
\end{aligned}$$

Let us introduce the abbreviations

$$\begin{aligned}
\mathcal{A} &:= -\frac{2}{\mathrm{Tr}(\mathrm{s})}\sum_{k,l=1}^{2}\left(\mathrm{s}_{M\times N}-\frac{1-\lambda_k^2}{1+\lambda_k^2}\mathrm{g}_{M\times N}\right)\Big(A(e_k,e_l),A(e_k,e_l)\Big), \\
\mathcal{C} &:= \frac{2}{\mathrm{Tr}(\mathrm{s})}\sum_{k,l=1}^{2}\left\{\frac{2\lambda_k^2}{1+\lambda_k^2}\mathrm{R}_M(e_k,e_l,e_k,e_l)-\frac{2}{1+\lambda_k^2}f^*\mathrm{R}_N(e_k,e_l,e_k,e_l)\right\}, \\
\mathcal{G} &:= \|\nabla\ln\mathrm{Tr}(\mathrm{s})\|^2.
\end{aligned}$$

Expanding A and $s_{M\times N}$ in terms of the normal basis $\{\xi_1, \xi_2\}$ and using equation (4.1.2), we obtain for the second fundamental form terms

$$\begin{aligned}
\mathcal{A} &= \frac{2}{\mathrm{Tr}(s)} \sum_{k,l,i=1}^{2} \frac{2(1-\lambda_i^2\lambda_k^2)}{(1+\lambda_i^2)(1+\lambda_k^2)} \Big(g_{M\times N}(A(e_k,e_l),\xi_i)\Big)^2 \\
&= \frac{2}{\mathrm{Tr}(s)} \sum_{k=1}^{2} \Bigg\{ \left(\frac{1-\lambda_1^2}{1+\lambda_1^2} + \frac{1-\lambda_2^2}{1+\lambda_2^2}\right) \Big(g_{M\times N}(A(e_1,e_k),\xi_2)\Big)^2 \\
&\qquad\qquad + \left(\frac{1-\lambda_1^2}{1+\lambda_1^2} + \frac{1-\lambda_2^2}{1+\lambda_2^2}\right) \Big(g_{M\times N}(A(e_2,e_k),\xi_1)\Big)^2 \\
&\qquad\qquad + 2\frac{1-\lambda_1^2}{1+\lambda_1^2}\Big(g_{M\times N}(A(e_1,e_k),\xi_1)\Big)^2 \\
&\qquad\qquad + 2\frac{1-\lambda_2^2}{1+\lambda_2^2}\Big(g_{M\times N}(A(e_2,e_k),\xi_2)\Big)^2 \Bigg\} \\
&= 2\sum_{k=1}^{2} \Big\{ \Big(g_{M\times N}(A(e_1,e_k),\xi_2)\Big)^2 + \Big(g_{M\times N}(A(e_2,e_k),\xi_1)\Big)^2 \Big\} \\
&\quad + \frac{4}{\mathrm{Tr}(s)} \sum_{k=1}^{2} \Bigg\{ \frac{1-\lambda_1^2}{1+\lambda_1^2}\Big(g_{M\times N}(A(e_1,e_k),\xi_1)\Big)^2 \\
&\qquad\qquad + \frac{1-\lambda_2^2}{1+\lambda_2^2}\Big(g_{M\times N}(A(e_2,e_k),\xi_2)\Big)^2 \Bigg\} .
\end{aligned}$$

The second sum may be written as

$$\begin{aligned}
&\frac{4}{\mathrm{Tr}(s)} \sum_{k=1}^{2} \Bigg\{ \frac{1-\lambda_1^2}{1+\lambda_1^2}\Big(g_{M\times N}(A(e_1,e_k),\xi_1)\Big)^2 \\
&\qquad + \frac{1-\lambda_2^2}{1+\lambda_2^2}\Big(g_{M\times N}(A(e_2,e_k),\xi_2)\Big)^2 \Bigg\} \\
&= 4\sum_{k=1}^{2} \Big\{ \Big(g_{M\times N}(A(e_1,e_k),\xi_1)\Big)^2 + \Big(g_{M\times N}(A(e_2,e_k),\xi_2)\Big)^2 \Big\} \\
&\quad - \frac{4}{\mathrm{Tr}(s)} \sum_{k=1}^{2} \Bigg\{ \frac{1-\lambda_2^2}{1+\lambda_2^2}\Big(g_{M\times N}(A(e_1,e_k),\xi_1)\Big)^2 \\
&\qquad\qquad + \frac{1-\lambda_1^2}{1+\lambda_1^2}\Big(g_{M\times N}(A(e_2,e_k),\xi_2)\Big)^2 \Bigg\} ,
\end{aligned}$$

so that we obtain the following expression,

$$
\begin{aligned}
\mathcal{A} = 2\sum_{k=1}^{2}&\left\{\Big(\mathrm{g}_{M\times N}(A(e_1,e_k),\xi_2)\Big)^2+\Big(\mathrm{g}_{M\times N}(A(e_2,e_k),\xi_1)\Big)^2\right\}\\
&+4\sum_{k=1}^{2}\left\{\Big(\mathrm{g}_{M\times N}(A(e_1,e_k),\xi_1)\Big)^2+\Big(\mathrm{g}_{M\times N}(A(e_2,e_k),\xi_2)\Big)^2\right\}\\
&-\frac{4}{\mathrm{Tr(s)}}\sum_{k=1}^{2}\left\{\frac{1-\lambda_2^2}{1+\lambda_2^2}\Big(\mathrm{g}_{M\times N}(A(e_1,e_k),\xi_1)\Big)^2\right.\\
&\qquad\left.+\frac{1-\lambda_1^2}{1+\lambda_1^2}\Big(\mathrm{g}_{M\times N}(A(e_2,e_k),\xi_2)\Big)^2\right\}\\
=2\|A\|^2&\\
&+2\left\{\Big(\mathrm{g}_{M\times N}(A(e_1,e_1),\xi_1)\Big)^2+\Big(\mathrm{g}_{M\times N}(A(e_1,e_2),\xi_1)\Big)^2\right.\\
&\quad\left.+\Big(\mathrm{g}_{M\times N}(A(e_1,e_2),\xi_2)\Big)^2+\Big(\mathrm{g}_{M\times N}(A(e_2,e_2),\xi_2)\Big)^2\right\}\\
&-\frac{4}{\mathrm{Tr(s)}}\sum_{k=1}^{2}\left\{\frac{1-\lambda_2^2}{1+\lambda_2^2}\Big(\mathrm{g}_{M\times N}(A(e_1,e_k),\xi_1)\Big)^2\right.\\
&\qquad\left.+\frac{1-\lambda_1^2}{1+\lambda_1^2}\Big(\mathrm{g}_{M\times N}(A(e_2,e_k),\xi_2)\Big)^2\right\}.
\end{aligned}
$$

Splitting the last summand and using the definition of the trace, we get

$$
\begin{aligned}
\mathcal{A} = 2\|A\|^2&+\frac{2}{\mathrm{Tr(s)}}\sum_{k=1}^{2}\left\{\frac{1-\lambda_1^2}{1+\lambda_1^2}\Big(\mathrm{g}_{M\times N}(A(e_1,e_k),\xi_1)\Big)^2\right.\\
&\qquad\left.+\frac{1-\lambda_2^2}{1+\lambda_2^2}\Big(\mathrm{g}_{M\times N}(A(e_2,e_k),\xi_2)\Big)^2\right\}\\
&-\frac{2}{\mathrm{Tr(s)}}\sum_{k=1}^{2}\left\{\frac{1-\lambda_2^2}{1+\lambda_2^2}\Big(\mathrm{g}_{M\times N}(A(e_1,e_k),\xi_1)\Big)^2\right.\\
&\qquad\left.+\frac{1-\lambda_1^2}{1+\lambda_1^2}\Big(\mathrm{g}_{M\times N}(A(e_2,e_k),\xi_2)\Big)^2\right\}\\
=2\|A\|^2&\\
&+\frac{2}{\mathrm{Tr(s)}}\left(\frac{1-\lambda_1^2}{1+\lambda_1^2}-\frac{1-\lambda_2^2}{1+\lambda_2^2}\right)\sum_{k=1}^{2}\left\{\Big(\mathrm{g}_{M\times N}(A(e_1,e_k),\xi_1)\Big)^2\right.\\
&\qquad\left.-\Big(\mathrm{g}_{M\times N}(A(e_2,e_k),\xi_2)\Big)^2\right\}.
\end{aligned}
$$

In the next step, let us consider the gradient terms. Note that

$$\mathcal{G} = \frac{\|\nabla \operatorname{Tr}(\mathrm{s})\|^2}{(\operatorname{Tr}(\mathrm{s}))^2}\,.$$

We are going to evaluate the norm in this expression. Using

$$(\nabla_{e_i}\mathrm{s})(e_j, e_j) = 2\mathrm{s}_{M\times N}(A(e_i, e_j), \mathrm{d}F(e_j))$$

and that the trace is parallel, we obtain

$$\begin{aligned}
\|\nabla \operatorname{Tr}(\mathrm{s})\|^2 &= \sum_{i=1}^{2}\left\{\sum_{j=1}^{2}(\nabla_{e_i}\mathrm{s})(e_j, e_j)\right\}^2 \\
&= 4\sum_{i=1}^{2}\Big\{\mathrm{s}_{M\times N}(A(e_i, e_1), \mathrm{d}F(e_1)) + \mathrm{s}_{M\times N}(A(e_i, e_2), \mathrm{d}F(e_2))\Big\}^2.
\end{aligned}$$

Since

$$\mathrm{s}_{M\times N}(A(e_i, e_j), \mathrm{d}F(e_j)) = -\frac{2\lambda_j}{1+\lambda_j^2}\mathrm{g}_{M\times N}(A(e_i, e_j), \xi_j)\,,$$

we calculate

$$\begin{aligned}
&\|\nabla \operatorname{Tr}(\mathrm{s})\|^2 \\
&= 4\sum_{i=1}^{2}\left\{\left(\frac{2\lambda_1}{1+\lambda_1^2}\right)^2\Big(\mathrm{g}_{M\times N}(A(e_1, e_i), \xi_1)\Big)^2\right. \\
&\qquad + \left(\frac{2\lambda_2}{1+\lambda_2^2}\right)^2\Big(\mathrm{g}_{M\times N}(A(e_2, e_i), \xi_2)\Big)^2 \\
&\qquad \left. + 2\frac{2\lambda_1}{1+\lambda_1^2}\frac{2\lambda_2}{1+\lambda_2^2}\mathrm{g}_{M\times N}(A(e_1, e_i), \xi_1)\mathrm{g}_{M\times N}(A(e_2, e_i), \xi_2)\right\},
\end{aligned}$$

so that

$$
\begin{aligned}
\mathcal{G} = {} & \frac{1}{2}\|\nabla \ln \mathrm{Tr}(\mathrm{s})\|^2 \\
& + \frac{2}{(\mathrm{Tr}(\mathrm{s}))^2} \sum_{i=1}^{2} \Bigg\{ \left(\frac{2\lambda_1}{1+\lambda_1^2}\right)^2 \Big(\mathrm{g}_{M\times N}(A(e_1,e_i),\xi_1)\Big)^2 \\
& \qquad + \left(\frac{2\lambda_2}{1+\lambda_2^2}\right)^2 \Big(\mathrm{g}_{M\times N}(A(e_2,e_i),\xi_2)\Big)^2 \\
& \qquad + 2\frac{2\lambda_1}{1+\lambda_1^2}\frac{2\lambda_2}{1+\lambda_2^2}\mathrm{g}_{M\times N}(A(e_1,e_i),\xi_1) \times \cdots \\
& \qquad\qquad \cdots \times \mathrm{g}_{M\times N}(A(e_2,e_i),\xi_2) \Bigg\} .
\end{aligned}
$$

Adding the last second fundamental form terms from $\mathcal{A}$ and the first two from $\mathcal{G}$,

$$
\begin{aligned}
& \frac{2}{(\mathrm{Tr}(\mathrm{s}))^2} \Bigg\{ \left(\frac{2\lambda_1}{1+\lambda_1^2}\right)^2 \Big(\mathrm{g}_{M\times N}(A(e_1,e_i),\xi_1)\Big)^2 \\
& \qquad + \left(\frac{2\lambda_2}{1+\lambda_2^2}\right)^2 \Big(\mathrm{g}_{M\times N}(A(e_2,e_i),\xi_2)\Big)^2 \\
& \quad + \mathrm{Tr}(\mathrm{s}) \left(\frac{1-\lambda_1^2}{1+\lambda_1^2} - \frac{1-\lambda_2^2}{1+\lambda_2^2}\right) \Big\{ \Big(\mathrm{g}_{M\times N}(A(e_1,e_i),\xi_1)\Big)^2 \\
& \qquad\qquad - \Big(\mathrm{g}_{M\times N}(A(e_2,e_i),\xi_2)\Big)^2 \Big\} \Bigg\} \\
& = \frac{2}{(\mathrm{Tr}(\mathrm{s}))^2} \Bigg\{ \frac{4\lambda_1^2}{(1+\lambda_1^2)^2} \Big(\mathrm{g}_{M\times N}(A(e_2,e_i),\xi_2)\Big)^2 \\
& \qquad + \frac{4\lambda_2^2}{(1+\lambda_2^2)^2} \Big(\mathrm{g}_{M\times N}(A(e_1,e_i),\xi_1)\Big)^2 \Bigg\} ,
\end{aligned}
$$

we may collect all terms to have in conclusion

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right) \ln \mathrm{Tr}(\mathrm{s}) \\
&= \mathcal{A} + \mathcal{G} + \mathcal{C} \\
&= 2\|A\|^2 + \frac{1}{2}\|\nabla \ln \mathrm{Tr}(\mathrm{s})\|^2 \\
&\quad + \frac{2}{(\mathrm{Tr}(\mathrm{s}))^2} \sum_{i=1}^{2} \left\{ \frac{2\lambda_2}{1+\lambda_2^2} \mathrm{g}_{M\times N}(A(e_1, e_i), \xi_1) \right. \\
&\qquad\qquad\qquad\qquad \left. + \frac{2\lambda_1}{1+\lambda_1^2} \mathrm{g}_{M\times N}(A(e_2, e_i), \xi_2) \right\}^2 \\
&\quad + \frac{2}{\mathrm{Tr}(\mathrm{s})} \sum_{k,l=1}^{2} \left\{ \frac{2\lambda_k^2}{1+\lambda_k^2} \mathrm{R}_M(e_k, e_l, e_k, e_l) \right. \\
&\qquad\qquad\qquad\qquad \left. - \frac{2}{1+\lambda_k^2} f^*\mathrm{R}_N(e_k, e_l, e_k, e_l) \right\},
\end{aligned}$$

which is the claim. □

Note that the non-curvature terms in equation (5.1.1) are non-negative. The next lemma decomposes the terms involving curvatures. This enables us to impose assumptions on the curvatures to conclude the non-negativity of these terms.

Lemma 5.1.2. *Let $\sigma \in \mathbb{R}$ be fixed. The curvature terms in equation* (5.1.1) *satisfy the relation*

$$\begin{aligned}
&\sum_{k,l=1}^{2} \left\{ \frac{2\lambda_k^2}{1+\lambda_k^2} \mathrm{R}_M(e_k, e_l, e_k, e_l) - \frac{2}{1+\lambda_k^2} f^*\mathrm{R}_N(e_k, e_l, e_k, e_l) \right\} \\
&\quad = \left(\frac{2\lambda_1^2}{1+\lambda_1^2} + \frac{2\lambda_2^2}{1+\lambda_2^2} \right) \frac{1}{1+\lambda_1^2} \frac{1}{1+\lambda_2^2} \Big(\sec_M(e_1 \wedge e_2) - \sigma \Big) \\
&\qquad - \left(\frac{2\lambda_1^2}{1+\lambda_1^2} + \frac{2\lambda_2^2}{1+\lambda_2^2} \right) \frac{\lambda_1^2}{1+\lambda_1^2} \frac{\lambda_2^2}{1+\lambda_2^2} \Big(\sec_N(\mathrm{d}f(e_1) \wedge \mathrm{d}f(e_2)) - \sigma \Big) \\
&\qquad - \mathrm{Tr}(\mathrm{s}) \frac{2\lambda_1^2\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)} \Big(\sec_N(\mathrm{d}f(e_1) \wedge \mathrm{d}f(e_2)) - \sigma \Big) \\
&\qquad + \mathrm{Tr}(\mathrm{s})\sigma \frac{\lambda_1^2 + \lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}.
\end{aligned}$$

Proof. We calculate

$$
\begin{aligned}
&\sum_{k,l=1}^{2}\left\{\frac{2\lambda_k^2}{1+\lambda_k^2}\mathrm{R}_M(e_k,e_l,e_k,e_l)-\frac{2}{1+\lambda_k^2}f^*\mathrm{R}_N(e_k,e_l,e_k,e_l)\right\}\\
&\quad=\left(\frac{2\lambda_1^2}{1+\lambda_1^2}+\frac{2\lambda_2^2}{1+\lambda_2^2}\right)\mathrm{R}_M(e_1,e_2,e_1,e_2)\\
&\qquad\qquad-\left(\frac{2}{1+\lambda_1^2}+\frac{2}{1+\lambda_2^2}\right)f^*\mathrm{R}_N(e_1,e_2,e_1,e_2)\\
&\quad=\left(\frac{2\lambda_1^2}{1+\lambda_1^2}+\frac{2\lambda_2^2}{1+\lambda_2^2}\right)\frac{1}{1+\lambda_1^2}\frac{1}{1+\lambda_2^2}\sec_M(e_1\wedge e_2)\\
&\qquad-\left(\frac{2}{1+\lambda_1^2}+\frac{2}{1+\lambda_2^2}\right)\frac{\lambda_1^2}{1+\lambda_1^2}\frac{\lambda_2^2}{1+\lambda_2^2}\sec_N(\mathrm{d}f(e_1)\wedge\mathrm{d}f(e_2))\\
&\quad=\left(\frac{2\lambda_1^2}{1+\lambda_1^2}+\frac{2\lambda_2^2}{1+\lambda_2^2}\right)\frac{1}{1+\lambda_1^2}\frac{1}{1+\lambda_2^2}\sec_M(e_1\wedge e_2)\\
&\qquad-2\left(\frac{1-\lambda_1^2}{1+\lambda_1^2}+\frac{1-\lambda_2^2}{1+\lambda_2^2}\right)\frac{\lambda_1^2}{1+\lambda_1^2}\frac{\lambda_2^2}{1+\lambda_2^2}\sec_N(\mathrm{d}f(e_1)\wedge\mathrm{d}f(e_2))\\
&\qquad-\left(\frac{2\lambda_1^2}{1+\lambda_1^2}+\frac{2\lambda_2^2}{1+\lambda_2^2}\right)\frac{\lambda_1^2}{1+\lambda_1^2}\frac{\lambda_2^2}{1+\lambda_2^2}\sec_N(\mathrm{d}f(e_1)\wedge\mathrm{d}f(e_2))\,.
\end{aligned}
$$

Using the definition of the trace,

$$
\mathrm{Tr}(\mathrm{s})=\frac{1-\lambda_1^2}{1+\lambda_1^2}+\frac{1-\lambda_2^2}{1+\lambda_2^2}\,,
$$

we further obtain

$$
\begin{aligned}
&\sum_{k,l=1}^{2}\left\{\frac{2\lambda_k^2}{1+\lambda_k^2}\mathrm{R}_M(e_k,e_l,e_k,e_l)-\frac{2}{1+\lambda_k^2}f^*\mathrm{R}_N(e_k,e_l,e_k,e_l)\right\}\\
&\quad=\left(\frac{2\lambda_1^2}{1+\lambda_1^2}+\frac{2\lambda_2^2}{1+\lambda_2^2}\right)\frac{1}{1+\lambda_1^2}\frac{1}{1+\lambda_2^2}\Big(\sec_M(e_1\wedge e_2)-\sigma\Big)\\
&\qquad-\left(\frac{2\lambda_1^2}{1+\lambda_1^2}+\frac{2\lambda_2^2}{1+\lambda_2^2}\right)\frac{\lambda_1^2}{1+\lambda_1^2}\frac{\lambda_2^2}{1+\lambda_2^2}\Big(\sec_N(\mathrm{d}f(e_1)\wedge\mathrm{d}f(e_2))-\sigma\Big)\\
&\qquad-2\,\mathrm{Tr}(\mathrm{s})\frac{\lambda_1^2}{1+\lambda_1^2}\frac{\lambda_2^2}{1+\lambda_2^2}\sec_N(\mathrm{d}f(e_1)\wedge\mathrm{d}f(e_2))\\
&\qquad+\frac{1}{1+\lambda_1^2}\frac{1}{1+\lambda_2^2}\left(\frac{2\lambda_1^2}{1+\lambda_1^2}+\frac{2\lambda_2^2}{1+\lambda_2^2}\right)\Big(1-\lambda_1^2\lambda_2^2\Big)\,\sigma\,.
\end{aligned}
$$

Using the definition of the trace again, we get

$$
\begin{aligned}
\sum_{k,l=1}^{2} & \left\{ \frac{2\lambda_k^2}{1+\lambda_k^2} \mathrm{R}_M(e_k, e_l, e_k, e_l) - \frac{2}{1+\lambda_k^2} f^*\mathrm{R}_N(e_k, e_l, e_k, e_l) \right\} \\
&= \left(\frac{2\lambda_1^2}{1+\lambda_1^2} + \frac{2\lambda_2^2}{1+\lambda_2^2} \right) \frac{1}{1+\lambda_1^2} \frac{1}{1+\lambda_2^2} \Big(\sec_M(e_1 \wedge e_2) - \sigma \Big) \\
&\quad - \left(\frac{2\lambda_1^2}{1+\lambda_1^2} + \frac{2\lambda_2^2}{1+\lambda_2^2} \right) \frac{\lambda_1^2}{1+\lambda_1^2} \frac{\lambda_2^2}{1+\lambda_2^2} \Big(\sec_N(\mathrm{d}f(e_1) \wedge \mathrm{d}f(e_2)) - \sigma \Big) \\
&\quad - 2\,\mathrm{Tr}(\mathrm{s}) \frac{\lambda_1^2}{1+\lambda_1^2} \frac{\lambda_2^2}{1+\lambda_2^2} \sec_N(\mathrm{d}f(e_1) \wedge \mathrm{d}f(e_2)) \\
&\quad + \left(\frac{\lambda_1^2}{1+\lambda_1^2} + \frac{\lambda_2^2}{1+\lambda_2^2} \right) \mathrm{Tr}(\mathrm{s}) \sigma .
\end{aligned}
$$

The last two terms simplify to

$$
\begin{aligned}
& \frac{\mathrm{Tr}(\mathrm{s})}{(1+\lambda_1^2)(1+\lambda_2^2)} \Big\{ \Big(\lambda_1^2 + \lambda_2^2 + 2\lambda_1^2\lambda_2^2 \Big) \sigma - 2\lambda_1^2\lambda_2^2 \sec_N \left(\mathrm{d}f(e_1) \wedge \mathrm{d}f(e_2)\right) \Big\} \\
&= \frac{\mathrm{Tr}(\mathrm{s})}{(1+\lambda_1^2)(1+\lambda_2^2)} \Big\{ \Big(\lambda_1^2 + \lambda_2^2 \Big) \sigma - 2\lambda_1^2\lambda_2^2 \left(\sec_N \left(\mathrm{d}f(e_1) \wedge \mathrm{d}f(e_2)\right) - \sigma \right) \Big\} .
\end{aligned}
$$

The claim follows by sorting the terms. □

Combining lemmas 5.1.1 and 5.1.2, we have shown the following.

Corollary 5.1.3. *Let* (M, g_M) *and* (N, g_N) *be Riemann surfaces. Under the mean curvature flow, the logarithm of the trace* $\mathrm{Tr}(\mathrm{s})$ *of* s *evolves by the equation*

$$
\begin{aligned}
& \left(\frac{\partial}{\partial t} - \Delta \right) \ln \mathrm{Tr}(\mathrm{s}) \\
&= 2\|A\|^2 + \frac{1}{2} \|\nabla \ln \mathrm{Tr}(\mathrm{s})\|^2 \\
&\quad + \frac{2}{(\mathrm{Tr}(\mathrm{s}))^2} \sum_{k=1}^{2} \Bigg\{ \frac{2\lambda_2}{1+\lambda_2^2} \mathrm{g}_{M\times N}(A(e_1, e_k), \xi_1) \\
&\qquad\qquad + \frac{2\lambda_1}{1+\lambda_1^2} \mathrm{g}_{M\times N}(A(e_2, e_k), \xi_2) \Bigg\}^2 \\
&\quad + \frac{2}{\mathrm{Tr}(\mathrm{s})} \left(\frac{2\lambda_1^2}{1+\lambda_1^2} + \frac{2\lambda_2^2}{1+\lambda_2^2} \right) \frac{1}{1+\lambda_1^2} \frac{1}{1+\lambda_2^2} \Big(\sec_M(e_1 \wedge e_2) - \sigma \Big) \\
&\quad + \frac{2}{\mathrm{Tr}(\mathrm{s})} \left(\frac{2\lambda_1^2}{1+\lambda_1^2} + \frac{2\lambda_2^2}{1+\lambda_2^2} \right) \frac{\lambda_1^2}{1+\lambda_1^2} \frac{\lambda_2^2}{1+\lambda_2^2} \Big(\sigma - \sec_N(\mathrm{d}f(e_1) \wedge \mathrm{d}f(e_2)) \Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4\lambda_1^2\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}\Big(\sigma-\sec_N(\mathrm{d}f(e_1)\wedge\mathrm{d}f(e_2))\Big)\\
&+2\sigma\frac{\lambda_1^2+\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}\,.
\end{aligned}
$$

As a direct consequence of the previous corollary, the following estimate holds.

Corollary 5.1.4. *Let* $\mathrm{Tr}(\mathrm{s})>0$ *and assume* $\sec_N\le\sigma\le\sec_M$. *Then the estimate*

$$
\left(\frac{\partial}{\partial t}-\Delta\right)\ln\mathrm{Tr}(\mathrm{s})\ge 2\|A\|^2+\frac{1}{2}\|\nabla\ln\mathrm{Tr}(\mathrm{s})\|^2+2\sigma\frac{\lambda_1^2+\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)} \tag{5.1.2}
$$

holds along the mean curvature flow.

The evolution inequality for $\ln(\mathrm{Tr}(\mathrm{s}))$ allows us to prove estimates for polynomials symmetric in the eigenvalues of the tensor s. Let us note the crucial observation, that the term containing the singular values in inequality (5.1.2) may be rewritten as

$$
\frac{\lambda_1^2+\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}=\frac{1}{2}\left(1-\frac{1-\lambda_1^2}{1+\lambda_1^2}\frac{1-\lambda_2^2}{1+\lambda_2^2}\right)=\frac{1}{2}\Big(1-\det(\mathrm{s})\Big)\,. \tag{5.1.3}
$$

Theorem 5.1.5. *Let M and N be Riemann surfaces, M being compact and N complete. Assume that there exists $\sigma\ge 0$, such that the sectional curvatures* $\sec_M$ *of M and* $\sec_N$ *of N satisfy the relation*

$$
\sec_N\le\sigma\le\sec_M\,.
$$

Then the following growth estimates hold for strictly area-decreasing maps along the mean curvature flow,

$$
\mathrm{Tr}(\mathrm{s})\ge 2\frac{\exp(\sigma t)}{\sqrt{c_1+\exp(2\sigma t)}}\quad\xrightarrow[t\to\infty]{\sigma>0}2\,, \tag{5.1.4}
$$

$$
\|\mathrm{s}\|^2\ge\frac{2}{1+c_1\exp(-2\sigma t)}\quad\xrightarrow[t\to\infty]{\sigma>0}2\,, \tag{5.1.5}
$$

$$
\det(\mathrm{s})\ge\frac{\exp(2\sigma t)-c_1}{\exp(2\sigma t)+c_1}\quad\xrightarrow[t\to\infty]{\sigma>0}1\,, \tag{5.1.6}
$$

$$
\det(\mathrm{s})\le\frac{1+2c_1\exp(-2\sigma t)}{1+c_1\exp(-2\sigma t)}\quad\xrightarrow[t\to\infty]{\sigma>0}1\,, \tag{5.1.7}
$$

where $c_1>0$ is a constant determined by the values of $\mathrm{Tr}(\mathrm{s})$ *on* $M\times\{0\}$.

Proof. Our idea is to apply the maximum principle (theorem 2.2.3) to inequality (5.1.2). By equation (5.1.3) we may express the singular values in terms of the determinant of s. Denote by s_{11} and s_{22} the eigenvalues of the (symmetric) tensor s. Then the relation

$$\det(s) = s_{11}s_{22} \le \frac{1}{2}s_{11}s_{22} + \frac{1}{4}\left(s_{11}^2 + s_{22}^2\right) = \frac{1}{4}\left(s_{11} + s_{22}\right)^2 = \frac{1}{4}\left(\mathrm{Tr}(s)\right)^2$$

holds, so we can estimate

$$\frac{\lambda_1^2 + \lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)} = \frac{1}{2}\left(1 - \det(s)\right) \ge \frac{1}{2}\left(1 - \frac{1}{4}\left(\mathrm{Tr}(s)\right)^2\right) = \frac{1}{8}\left(4 - \left(\mathrm{Tr}(s)\right)^2\right).$$

Using that $\sigma \ge 0$ by assumption, we can estimate inequality (5.1.2) by

$$\left(\frac{\partial}{\partial t} - \Delta\right)\ln \mathrm{Tr}(s) \ge \frac{\sigma}{4}\left(4 - \left(\mathrm{Tr}(s)\right)^2\right).$$

The solution (for $\inf_{M\times\{0\}} \mathrm{Tr}(s) > 0$) of the associated ordinary differential equation

$$\frac{\partial}{\partial t}\ln u = \frac{\sigma}{4}\left(4 - u^2\right), \qquad u(0) := \inf_{M\times\{0\}} \mathrm{Tr}(s),$$

is given by (see appendix A.1)

$$u(t) = \frac{2\exp(\sigma t)}{\sqrt{c_1 + \exp(2\sigma t)}}, \qquad c_1 := \frac{4}{(u(0))^2} - 1 > 0.$$

Applying theorem 2.2.2 establishes inequality (5.1.4).

Inequality (5.1.5) follows from the inequality $\left(\mathrm{Tr}(s)\right)^2 \le 2\|s\|^2$. The inequalities (5.1.6) and (5.1.7) follow from $\det(s) = \frac{1}{2}\{(\mathrm{Tr}(s))^2 - \|s\|^2\}$, $\mathrm{Tr}(s) \le 2$ and $\|s\|^2 \le 2$. □

Remark 5.1.6. In the case $\sigma = 0$, theorem 5.1.5 only states the boundedness of the trace, the norm and the determinant of s. Explicitly, setting $u_0 := \inf_{p\in M}\mathrm{Tr}(s)$, in this case we have

$$\mathrm{Tr}(s) \ge \frac{2}{\sqrt{c_1+1}} = u_0, \qquad \|s\|^2 \ge \frac{2}{1+c_1} = \frac{u_0^2}{2},$$

$$\det(s) \ge \frac{1-c_1}{1+c_1} = \frac{u_0^2}{2} - 1, \qquad \det(s) \le \frac{1+2c_1}{1+c_1} = 2 - \frac{u_0^2}{4}.$$

Remark 5.1.7. If $c_1 \le 1$, the determinant is non-negative along the flow. If $c_1 > 1$, the determinant is non-negative at least for $2\sigma t \ge \ln c_1$. Unfortunately, this does not yield any information in the case $\sigma = 0$.

Remark 5.1.8. With the same technique used to obtain the estimates for the trace of s, we also obtain an estimate for associated Jacobian $\mathrm{Jac}(\pi_M)$ of the projection map $\pi_M : F(M) \to M$, which is defined by the relation

$$\pi_M^* \Omega_M = \mathrm{Jac}(\pi_M) \Omega_{F(M)} ,$$

where Ω_M denotes the volume form on (M, g_M) and $\Omega_{F(M)}$ denotes the volume form on $F(M)$ coming from the induced metric g. Let us set $\Omega := (\pi_M \circ F)^* \Omega_M$. Then the associated Jacobian may equivalently be expressed as

$$v := \mathrm{Jac}(\pi_M) = \star \Omega ,$$

where $\star : \Omega^k(M) \to \Omega^{2-k}(M)$ is the Hodge star operator with respect to the induced metric g.

Consider a weakly area-decreasing map $f : M \to N$. Assume that the weakly area-decreasing property is preserved under the mean curvature flow, i. e. it holds $\lambda_1^2 \lambda_2^2 \leq 1$ for all $(x,t) \in M \times [0,T)$. Fix a point $(x_0, t_0) \in M \times [0,T)$. In terms of the singular values λ_1^2 and λ_2^2, $v(x_0,t_0)$ is given by

$$v(x_0,t_0) = \frac{1}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}} .$$

The evolution equation for v is essentially derived in [Wan02, Proposition 3.2]. By [SS14c, Lemma 3.4], the evolution equation for $\ln v$ reads

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) \ln v = \|A\|^2 &+ \lambda_1^2 \Big((A_{11}^1)^2 + (A_{12}^1)^2 \Big) + \lambda_2^2 \Big((A_{12}^2)^2 + (A_{22}^2)^2 \Big) \\ &+ 2\lambda_1 \lambda_2 \Big(A_{11}^2 A_{21}^1 + A_{12}^2 A_{22}^1 \Big) \\ &+ \frac{1}{(1+\lambda_1^2)(1+\lambda_2^2)} \Big\{ (\lambda_1^2 + \lambda_2^2) \sec_M - 2\lambda_1^2\lambda_2^2 \sec_N \Big\} .\end{aligned} \tag{5.1.8}$$

The curvature terms may be rewritten as

$$\begin{aligned}(\lambda_1^2 + \lambda_2^2) \sec_M - 2\lambda_1^2\lambda_2^2 \sec_N = (\lambda_1^2 + \lambda_2^2)(\sec_M - \sigma) &+ 2\lambda_1^2\lambda_2^2 (\sigma - \sec_N) \\ &+ (\lambda_1^2 + \lambda_2^2 - 2\lambda_1^2\lambda_2^2)\sigma .\end{aligned}$$

Let us impose the curvature assumptions $\sec_N \leq \sigma \leq \sec_M$ for some $\sigma \geq 0$. Then the first two terms are non-negative, so that

$$(\lambda_1^2 + \lambda_2^2) \sec_M - 2\lambda_1^2\lambda_2^2 \sec_N \geq (\lambda_1^2 + \lambda_2^2 - 2\lambda_1^2\lambda_2^2)\sigma .$$

For the second fundamental form terms in equation (5.1.8), we carry out a similar calculation as in [SS14c, Lemma 3.5]. Since by assumption $\lambda_1^2\lambda_2^2 \leq 1$, for any

$t \in [0, T)$ there exists $\delta \in [0, 1]$ with $\lambda_1^2 \lambda_2^2 \leq 1 - \delta$. Then

$$\begin{aligned}
&\|A\|^2 + 2\lambda_1\lambda_2\Big(A^2_{11}A^1_{21} + A^2_{12}A^1_{22}\Big) \\
&\quad \geq \delta\|A\|^2 + (1-\delta)\|A\|^2 - 2(1-\delta)\Big(|A^2_{11}A^1_{21}| + |A^2_{12}A^1_{22}|\Big) \\
&\quad \geq \delta\|A\|^2 + \frac{1}{4}(1-\delta)\|A\|^2 + (1-\delta)\left(\sqrt{\frac{2}{3}}|A^2_{11}| - \sqrt{\frac{3}{2}}|A^1_{21}|\right)^2 \\
&\qquad\qquad + (1-\delta)\left(\sqrt{\frac{3}{2}}|A^2_{12}| - \sqrt{\frac{2}{3}}|A^1_{22}|\right)^2 \\
&\quad \geq \frac{1}{4}\|A\|^2 .
\end{aligned}$$

In conclusion, the calculations yield the estimate

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) \ln v \quad &\geq \quad \frac{1}{4}\|A\|^2 + \frac{\lambda_1^2 + \lambda_2^2 - 2\lambda_1^2\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}\sigma \\
&\geq \quad \left(1 - \frac{1 + 3\lambda_1^2\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}\right)\sigma \\
&\overset{\lambda_1^2\lambda_2^2 \leq 1}{\geq} \quad (1 - 4v^2)\sigma .
\end{aligned}$$

The (positive) solution to the associated ordinary differential equation

$$\frac{\partial}{\partial t}\ln w = (1 - 4w^2)\sigma , \qquad w(0) = \inf_{M\times\{0\}}(\star\Omega)$$

is given by

$$w(t) = \frac{1}{\sqrt{4 + c_1\exp(-2\sigma t)}} , \qquad \text{with} \qquad c_1 := \frac{1}{\Big(\inf_{M\times\{0\}}(\star\Omega)\Big)^2} - 4 .$$

Applying the weak maximum principle (theorem 2.2.3), we therefore have shown

$$\star\Omega = v \geq w(t) = \frac{1}{\sqrt{4 + c_1\exp(-2\sigma t)}}$$

for every map $f : M \to N$ which stays weakly area-decreasing under the mean curvature flow on the interval of existence of the flow. In particular, if we assume $\sigma > 0$, then the function w converges to

$$\lim_{t\to\infty} w(t) = \frac{1}{2} ,$$

which is consistent with weakly area-decreasing maps converging to isometries (where we have $\star\Omega \overset{t\to\infty}{\longrightarrow} \frac{1}{2}$) or constant maps (in this case it is $\star\Omega \overset{t\to\infty}{\longrightarrow} 1$) [Wan02; Wan05; TW04; LL11; SS14c]. At this point, if we have the additional information that the map is area-preserving and remains so (e. g. if it is a symplectomorphism from S^2 to itself, see [Wan01; MW11]), the estimates imply the convergence to an isometry.

Considering the relation

$$4\,(\star\Omega)^2 = 1 + \mathrm{Tr}(\mathrm{s}) + \det(\mathrm{s})$$

and using the estimates from theorem 5.1.5, for strictly area-decreasing maps we obtain another explicit growth estimate for the Jacobian of the projection π_M. In particular,

$$\star\Omega \quad \overset{\sigma>0}{\underset{}{\overset{t\to\infty}{\longrightarrow}}} \quad 1\,.$$

Note that yet another growth estimate was derived in [LL11, Proof of Theorem 2 (ii)]. Following the proof outlined in this paper, one can apply the methods from [Wan02] to show smooth convergence to a constant map.

5.2 A Decay Estimate for the Mean Curvature Vector

We want to use the estimates for the tensor s proven in theorem 5.1.5 to control the norm of the mean curvature vector. In order to do this, note that the evolution inequality for $\ln \mathrm{Tr}(\mathrm{s})$ implies an evolution inequality for the following modified quantity.

Lemma 5.2.1. *Let $h : [0, T) \to \mathbb{R}^{>0}$ be an arbitrary differentiable positive function. Then the following evolution inequality holds,*

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) \ln\Big(h(t)\,\mathrm{Tr}(\mathrm{s})\Big) \geq 2\|A\|^2 + \frac{1}{2}\|\nabla \ln(h(t)\,\mathrm{Tr}(\mathrm{s}))\|^2 \\ + 2\sigma\frac{\lambda_1^2+\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)} + \frac{h'(t)}{h(t)}\,.\end{aligned}$$

Proof. This follows directly from corollary 5.1.4. □

Following the ideas presented [STW14], our aim now is to use the evolution inequality (3.2.1) for the norm of $\vec{H}$ and the evolution inequality for $\ln(h(t)\,\mathrm{Tr}(\mathrm{s}))$ as given by lemma 5.2.1 to get the desired estimate on $\|\vec{H}\|^2$. In particular, we need to estimate the curvature terms in inequality (3.2.1) which involve the mean curvature vector. For that purpose, we first decompose the curvature terms. The resulting decomposition then may be estimated by known and more usable terms.

Lemma 5.2.2. *Let (M, g_M) and (N, g_N) be Riemann surfaces. The curvature tensor occurring in inequality* (3.2.1) *may be decomposed as*

$$\sum_{k=1}^{2} \mathrm{R}_{M\times N}(\vec{H}, \mathrm{d}F(e_k), \vec{H}, \mathrm{d}F(e_k)) \\ = \frac{1}{(1+\lambda_1^2)(1+\lambda_2^2)} \Big\{ \Big((H^1)^2\lambda_1^2 + (H^2)^2\lambda_2^2 \Big) \sec_M \\ + \Big((H^2)^2\lambda_1^2 + (H^1)^2\lambda_2^2 \Big) \sec_N \Big\},$$

where $H^k := g_{M\times N}(\vec{H}, \xi_k)$.

Proof. Recall that $\mathrm{R}_{M\times N} = \pi_M^* \mathrm{R}_M \oplus \pi_N^* \mathrm{R}_N$ and set (using the basis defined in section 2.3)

$$(\mathrm{R}_M)_{ijkl} := \mathrm{R}_M(\alpha_i, \alpha_j, \alpha_k, \alpha_l), \qquad (\mathrm{R}_N)_{ijkl} := \mathrm{R}_N(\beta_i, \beta_j, \beta_k, \beta_l).$$

We calculate

$$\mathrm{R}_{M\times N}(\vec{H}, \mathrm{d}F(e_k), \vec{H}, \mathrm{d}F(e_k)) \\ = \sum_{i,j=1}^{2} \mathrm{R}_{M\times N}(H^i\xi_i, \mathrm{d}F(e_k), H^j\xi_j, \mathrm{d}F(e_k)) \\ = \sum_{i,j=1}^{2} H^iH^j(\pi_M^*\mathrm{R}_M \oplus \pi_N^*\mathrm{R}_N)(\xi_i, \mathrm{d}F(e_k), \xi_j, \mathrm{d}F(e_k)) \\ = \sum_{i,j=1}^{2} H^iH^j \frac{\lambda_i\lambda_j}{\sqrt{(1+\lambda_i^2)(1+\lambda_j^2)}} \frac{1}{1+\lambda_k^2} (\mathrm{R}_M)_{ikjk} \\ + \sum_{i,j=1}^{2} H^iH^j \frac{1}{\sqrt{(1+\lambda_i^2)(1+\lambda_j^2)}} \frac{\lambda_k^2}{1+\lambda_k^2} (\mathrm{R}_N)_{ikjk}.$$

Setting $k := 1$ and using the symmetries of the Riemannian curvature tensors, we obtain

$$\mathrm{R}_{M\times N}(\vec{H}, \mathrm{d}F(e_1), \vec{H}, \mathrm{d}F(e_1)) \\ = \sum_{i,j=1}^{2} H^iH^j \frac{\lambda_i\lambda_j}{\sqrt{(1+\lambda_i^2)(1+\lambda_j^2)}} \frac{1}{1+\lambda_1^2} (\mathrm{R}_M)_{i1j1} \\ + \sum_{i,j=1}^{2} H^iH^j \frac{1}{\sqrt{(1+\lambda_i^2)(1+\lambda_j^2)}} \frac{\lambda_1^2}{1+\lambda_1^2} (\mathrm{R}_N)_{i1j1} \\ = \frac{(H^2)^2}{(1+\lambda_1^2)(1+\lambda_2^2)} \Big(\lambda_2^2(\mathrm{R}_M)_{2121} + \lambda_1^2(\mathrm{R}_N)_{2121} \Big)$$

$$= \frac{(H^2)^2}{(1+\lambda_1^2)(1+\lambda_2^2)}\left(\lambda_2^2 \sec_M + \lambda_1^2 \sec_N\right).$$

In the same way we get

$$\mathrm{R}_{M\times N}(\vec{H}, \mathrm{d}F(e_2), \vec{H}, \mathrm{d}F(e_2)) = \frac{(H^1)^2}{(1+\lambda_1^2)(1+\lambda_2^2)}\left(\lambda_1^2 \sec_M + \lambda_2^2 \sec_N\right).$$

Summing up both equations, we obtain the claim. □

Corollary 5.2.3. *Let (M, g_M) and (N, g_N) be Riemann surfaces and assume there exist constants $\sigma, \kappa_M \geq 0$, such that $\sec_N \leq \sigma$ and $\sec_M \leq \kappa_M$. Then the estimate*

$$\sum_{k=1}^{2} \mathrm{R}_{M\times N}(\vec{H}, \mathrm{d}F(e_k), \vec{H}, \mathrm{d}F(e_k)) \leq \|\vec{H}\|^2 \frac{\lambda_1^2+\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}(\kappa_M + \sigma)$$

holds. Further, for $\varepsilon, \delta > 0$ with $\delta \leq \varepsilon$, the evolution inequality

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) \ln(\delta t\|\vec{H}\|^2 + \varepsilon) &\leq 2\|A\|^2 + \frac{1}{2}\|\nabla(\delta t\|\vec{H}\|^2+\varepsilon)\|^2 \\ &\quad + 2\frac{\delta t\|\vec{H}\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}\frac{\lambda_1^2+\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}(\kappa_M+\sigma)\end{aligned}$$

is satisfied.

Proof. The first claim is a direct consequence of lemma 5.2.2. The second claim follows by the proof for [STW14, Lemma 3.2], which, for completeness, we state here. Note that we carry out the calculation for arbitrary $m = \dim M$ and then specialize to the two-dimensional case. The evolution equation for $\ln(\delta t\|\vec{H}\|^2 + \varepsilon)$ in terms of the evolution equation for $\|\vec{H}\|^2$ is given by

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) \ln(\delta t\|\vec{H}\|^2 + \varepsilon) &= \frac{\delta t}{\delta t\|\vec{H}\|^2+\varepsilon}\left(\frac{\partial}{\partial t} - \Delta\right)\|\vec{H}\|^2 \\ &\quad + \frac{\delta\|\vec{H}\|^2}{\delta t\|\vec{H}\|^2+\varepsilon} + \|\nabla \ln(\delta t\|\vec{H}\|^2+\varepsilon)\|^2 .\end{aligned}$$

Using inequality (3.2.1), we obtain the estimate

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)\ln(\delta t\|\vec{H}\|^2+\varepsilon)\\
&\quad\le \frac{\delta t}{\delta t\|\vec{H}\|^2+\varepsilon}\Big\{-2\|\nabla^\perp\vec{H}\|^2+2\|\vec{H}\|^2\|A\|^2\\
&\qquad\qquad\qquad\qquad\qquad\qquad +2\sum_{k=1}^m \mathrm{R}_{M\times N}(\vec{H},\mathrm{d}F(e_k),\vec{H},\mathrm{d}F(e_k))\Big\}\\
&\qquad\qquad +\frac{\delta\|\vec{H}\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}+\|\nabla\ln(\delta t\|\vec{H}\|^2+\varepsilon)\|^2\\
&\quad=\frac{2\delta t\|\vec{H}\|^2\|A\|^2+2\varepsilon\|A\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}-\frac{2\varepsilon\|A\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}-\frac{2\delta t\|\nabla^\perp\vec{H}\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}\\
&\qquad\qquad +\frac{2\delta t}{\delta t\|\vec{H}\|^2+\varepsilon}\sum_{k=1}^m \mathrm{R}_{M\times N}(\vec{H},\mathrm{d}F(e_k),\vec{H},\mathrm{d}F(e_k))\\
&\qquad\qquad +\frac{\delta\|\vec{H}\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}+\|\nabla\ln(\delta t\|\vec{H}\|^2+\varepsilon)\|^2\,.
\end{aligned}
$$

To go on, note that the Cauchy-Schwarz inequality implies

$$
\|\nabla\|\vec{H}\|^2\|^2=\|2\langle\nabla^\perp\vec{H},\vec{H}\rangle\|^2\le 4\|\nabla^\perp\vec{H}\|^2\|\vec{H}\|^2\,,
$$

which we use to estimate

$$
\begin{aligned}
&-\frac{2\delta t\|\nabla^\perp\vec{H}\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}+\frac{1}{2}\|\nabla\ln(\delta t\|\vec{H}\|^2+\varepsilon)\|^2\\
&\quad=-\frac{2\delta t\|\nabla^\perp\vec{H}\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}+\frac{1}{2}\frac{\delta^2t^2\|\nabla\|\vec{H}\|^2\|^2}{(\delta t\|\vec{H}\|^2+\varepsilon)^2}\\
&\quad\le-\frac{2\delta t\|\nabla^\perp\vec{H}\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}+\frac{2\delta^2t^2\|\nabla^\perp\vec{H}\|^2\|\vec{H}\|^2}{(\delta t\|\vec{H}\|^2+\varepsilon)^2}\\
&\quad=-\frac{2\delta t\|\nabla^\perp\vec{H}\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}\left(1-\frac{\delta t\|\vec{H}\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}\right)\\
&\quad=-\frac{2\delta t\|\nabla^\perp\vec{H}\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}\frac{\varepsilon}{\delta t\|\vec{H}\|^2+\varepsilon}\\
&\quad\le 0\,.
\end{aligned}
$$

Therefore, we obtain

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-\Delta\right)\ln(\delta t\|\vec{H}\|^2+\varepsilon)\\&\leq 2\|A\|^2-\frac{2\varepsilon\|A\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}+\frac{\delta\|\vec{H}\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}+\frac{1}{2}\|\nabla\ln(\delta t\|\vec{H}\|^2+\varepsilon)\|^2\\&\qquad+\frac{2\delta t}{\delta t\|\vec{H}\|^2+\varepsilon}\sum_{k=1}^{m}\mathrm{R}_{M\times N}(\vec{H},\mathrm{d}F(e_k),\vec{H},\mathrm{d}F(e_k))\,.\end{aligned}$$

Since $\|\vec{H}\|^2\leq m\|A\|^2$, we further estimate

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-\Delta\right)\ln(\delta t\|\vec{H}\|^2+\varepsilon)\\&\leq 2\|A\|^2+\frac{(\delta m-2\varepsilon)\|A\|^2}{\delta t\|\vec{H}\|^2+\varepsilon}+\frac{1}{2}\|\nabla\ln(\delta t\|\vec{H}\|^2+\varepsilon)\|^2\\&\qquad+\frac{2\delta t}{\delta t\|\vec{H}\|^2+\varepsilon}\sum_{k=1}^{m}\mathrm{R}_{M\times N}(\vec{H},\mathrm{d}F(e_k),\vec{H},\mathrm{d}F(e_k))\,.\end{aligned}$$

Choosing $m=2$, $\delta\leq\frac{2\varepsilon}{m}=\varepsilon$ and using the estimate for the curvature terms, we obtain the claim. □

At this point we have the necessary prerequisites available to prove the following statement.

Theorem 5.2.4. *Let M and N be Riemann surfaces, M being compact and N complete. Assume that there exists $\sigma\geq 0$, such that the sectional curvatures $\sec_M$ of M and $\sec_N$ of N satisfy the relation*

$$\sec_N\leq\sigma\leq\sec_M\ .$$

Let $f: M\to N$ be a smooth map and evolve it by mean curvature flow. If the initial map f_0 is strictly area-decreasing, then each $F_t(M)$ is the graph of some strictly area-decreasing map f_t. Further, if $\sigma>0$, the mean curvature vector satisfies the estimate

$$t\|\vec{H}\|^2\leq C$$

along the mean curvature flow for some constant $C\geq 0$, which depends on the initial values of $\mathrm{Tr}(\mathrm{s})$, σ and on the maximum of the sectional curvatures $\kappa_M:=\max_{p\in M}\sec_M(p)$ of M.

If N is compact and $\sigma>0$, by [LL11, Theorem 2] (or, more recently, [SS14c, Theorem A]) the assumptions of theorem 5.2.4 imply the smooth convergence to a constant map.

Proof. Using the assumptions $\sec_N \leq \sigma \leq \sec_M$ for some $\sigma \geq 0$ and $\mathrm{Tr}(\mathrm{s}) > 0$, from corollary 5.1.4 and the maximum principle (theorem 2.2.3) we obtain that the lower bound of $\mathrm{Tr}(\mathrm{s})$ is preserved under the mean curvature flow, so that

$$\mathrm{Tr}(\mathrm{s}) \geq \inf_{M\times\{0\}} \mathrm{Tr}(\mathrm{s}) > 0\,.$$

In particular, any $F_t(M)$ is the graph of a strictly area-decreasing map f_t.

Now assume $\sigma > 0$. Using the evolution inequality for $\ln(h(t)\,\mathrm{Tr}(\mathrm{s}))$ (see lemma 5.2.1) and the evolution inequality for $\ln(\delta t\|\vec{H}\|^2 + \varepsilon)$ (see corollary 5.2.3) together with

$$\frac{\delta t\|\vec{H}\|^2}{\delta t\|\vec{H}\|^2 + \varepsilon} \leq 1\,,$$

we calculate

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right) \ln \frac{\delta t\|\vec{H}\|^2 + \varepsilon}{h(t)\,\mathrm{Tr}(\mathrm{s})} \\
&\quad\leq \quad \frac{1}{2}\|\nabla \ln(\delta t\|\vec{H}\|^2 + \varepsilon)\|^2 - \frac{1}{2}\|\nabla \ln(h(t)\,\mathrm{Tr}(\mathrm{s}))\|^2 \\
&\qquad\qquad + 2\frac{\lambda_1^2 + \lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}\kappa_M - \frac{h'(t)}{h(t)} \\
&\quad= \quad \frac{1}{2}\left\langle \nabla \ln\Big((\delta t\|\vec{H}\|^2 + \varepsilon)h(t)\,\mathrm{Tr}(\mathrm{s})\Big), \nabla \ln \frac{\delta t\|\vec{H}\|^2 + \varepsilon}{h(t)\,\mathrm{Tr}(\mathrm{s})}\right\rangle \\
&\qquad\qquad + \kappa_M(1 - \det(\mathrm{s})) - \frac{h'(t)}{h(t)} \\
&\quad\overset{\text{Eq. (5.1.6)}}{\leq} \quad \frac{1}{2}\left\langle \nabla \ln\Big((\delta t\|\vec{H}\|^2 + \varepsilon)h(t)\,\mathrm{Tr}(\mathrm{s})\Big), \nabla \ln \frac{\delta t\|\vec{H}\|^2 + \varepsilon}{h(t)\,\mathrm{Tr}(\mathrm{s})}\right\rangle \\
&\qquad\qquad + \kappa_M\left(1 - \frac{\exp(2\sigma t) - c_1}{\exp(2\sigma t) + c_1}\right) - \frac{h'(t)}{h(t)}\,,
\end{aligned}$$

where the (positive) function h remains to be chosen. To do this, consider the ordinary differential equation

$$\frac{h'(t)}{h(t)} = \kappa_M\left(1 - \frac{\exp(2\sigma t) - c_1}{\exp(2\sigma t) + c_1}\right)\,.$$

Since $\sigma > 0$, the solution to this equation is given by (see appendix A.2)

$$h(t) = c_2\frac{\exp(2\kappa_M t)}{(c_1 + \exp(2\sigma t))^{\kappa_M/\sigma}}\,, \qquad c_2 > 0\,.$$

Using this particular h, we obtain

$$\left(\frac{\partial}{\partial t} - \Delta\right) \ln \frac{\delta t\|\vec{H}\|^2 + \varepsilon}{h(t)\,\mathrm{Tr}(\mathrm{s})} \leq \frac{1}{2} \left\langle \nabla \ln\Big((\delta t\|\vec{H}\|^2 + \varepsilon)h(t)\,\mathrm{Tr}(\mathrm{s})\Big), \nabla \ln \frac{\delta t\|\vec{H}\|^2 + \varepsilon}{h(t)\,\mathrm{Tr}(\mathrm{s})} \right\rangle$$

and therefore establish the boundedness of the quantity on the left-hand side by applying the maximum principle (theorem 2.2.3). Note that

$$h'(t) = 2c_2\kappa_M h(t) \left\{1 - \frac{\exp(2\sigma t)}{c_1 + \exp(2\sigma t)}\right\} \overset{c_1>0}{\geq} 0\,,$$

so that h is monotonically increasing. Further,

$$\lim_{t\to\infty} h(t) = c_2\,.$$

Since we are free to choose $0 < \delta \leq \varepsilon$ and $c_2 > 0$, we set them to $\varepsilon := \delta := c_2 := 1$. Then, by the maximum principle,

$$\frac{t\|\vec{H}\|^2 + 1}{h(t)\,\mathrm{Tr}(\mathrm{s})} \leq \sup_{M\times\{0\}} \frac{t\|\vec{H}\|^2 + 1}{h(t)\,\mathrm{Tr}(\mathrm{s})} = (1 + c_1)^{\kappa_M/\sigma} \sup_{M\times\{0\}} \frac{1}{\mathrm{Tr}(\mathrm{s})} =: C(c_1, \kappa_M, \sigma)$$

along the mean curvature flow. Rearranging the terms, we obtain

$$t\|\vec{H}\|^2 \leq C(c_1, \kappa_M, \sigma)h(t)\,\mathrm{Tr}(\mathrm{s}) - 1 \leq 2C(c_1, \kappa_M, \sigma) - 1\,.$$

This shows the theorem. □

Remark 5.2.5. Note that if $\kappa_M = 0$, we have $\sec_N \leq 0 = \sigma = \sec_M$. Now the curvature term in the evolution equation for $\|\vec{H}\|^2$ is automatically non-positive (see corollary 5.2.3), so that the proof given in [STW14] extends to this case.

5.3 A Decay Estimate for the Second Fundamental Form

Having established a decay estimate for the mean curvature vector, we want to prove a similar estimate for the second fundamental form. In order to achieve this, we first derive an evolution equation for an arbitrary function of the trace of s. This enables us to obtain a large coefficient of $\|A\|^2$ on the right hand side of the resulting equation. This term then may be used to get a better evolution equation for the second fundamental form.

Lemma 5.3.1. *Let (M, g_M) and (N, g_N) be Riemann surfaces and assume that there exists $\sigma \geq 0$, such that $\sec_N \leq \sigma \leq \sec_M$. Let $p : [0,2] \to \mathbb{R}^{>0}$ be a smooth function. If $p'(x) > 0$, the differential inequality*

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) \ln p(\mathrm{Tr}(\mathrm{s})) \geq{}& 2\frac{p'}{p} \mathrm{Tr}(\mathrm{s}) \|A\|^2 + \frac{1}{2} \|\nabla \ln p(\mathrm{Tr}(\mathrm{s}))\|^2 \\ &- \left(\frac{1}{2}\frac{p'}{p}\frac{1}{\mathrm{Tr}(\mathrm{s})} + \frac{p''}{p} - \frac{1}{2}\frac{(p')^2}{p^2}\right) \|\nabla \mathrm{Tr}(\mathrm{s})\|^2 \\ &+ 2\sigma \frac{p'}{p} \mathrm{Tr}(\mathrm{s}) \frac{\lambda_1^2 + \lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}\end{aligned}$$

holds under the mean curvature flow.

Proof. We calculate

$$\left(\frac{\partial}{\partial t} - \Delta\right) \ln p(\mathrm{Tr}(\mathrm{s})) = \frac{p'}{p}\left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{Tr}(\mathrm{s}) - \frac{(p'')p - (p')^2}{p^2} \|\nabla \mathrm{Tr}(\mathrm{s})\|^2 . \quad (5.3.1)$$

Setting $p(x) := x$, it follows that

$$\left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{Tr}(\mathrm{s}) = \mathrm{Tr}(\mathrm{s}) \left(\frac{\partial}{\partial t} - \Delta\right) \ln \mathrm{Tr}(\mathrm{s}) - \frac{\|\nabla \mathrm{Tr}(\mathrm{s})\|^2}{\mathrm{Tr}(\mathrm{s})} .$$

Inserting this into equation (5.3.1), we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) \ln p(\mathrm{Tr}(\mathrm{s})) ={}& \frac{p'}{p} \mathrm{Tr}(\mathrm{s}) \left(\frac{\partial}{\partial t} - \Delta\right) \ln \mathrm{Tr}(\mathrm{s}) - \frac{p'}{p}\frac{\|\nabla \mathrm{Tr}(\mathrm{s})\|^2}{\mathrm{Tr}(\mathrm{s})} \\ &- \frac{p''}{p} \|\nabla \mathrm{Tr}(\mathrm{s})\|^2 + \|\nabla \ln p(\mathrm{Tr}(\mathrm{s}))\|^2 .\end{aligned}$$

Using the evolution inequality (5.1.2) for $\ln \mathrm{Tr}(\mathrm{s})$ and the assumption $p' > 0$, we further have

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \Delta\right) \ln p(\mathrm{Tr}(\mathrm{s})) \\ &\quad\geq 2\frac{p'}{p} \mathrm{Tr}(\mathrm{s}) \|A\|^2 + \frac{1}{2} \|\nabla \ln p(\mathrm{Tr}(\mathrm{s}))\|^2 \\ &\qquad + \left(\frac{1}{2}\frac{p'}{p}\frac{1}{\mathrm{Tr}(\mathrm{s})} - \frac{p'}{p}\frac{1}{\mathrm{Tr}(\mathrm{s})} - \frac{p''}{p} + \frac{1}{2}\frac{(p')^2}{p^2}\right) \|\nabla \mathrm{Tr}(\mathrm{s})\|^2 \\ &\qquad + 2\sigma \frac{p'}{p} \mathrm{Tr}(\mathrm{s}) \frac{\lambda_1^2 + \lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)} .\end{aligned}$$

This shows the claim. □

Consider the differential inequality which follows from demanding the coefficient of $\|\nabla \operatorname{Tr}(s)\|^2$ to be non-negative,

$$\frac{1}{2}\frac{p'}{xp} + \frac{p''}{p} - \frac{1}{2}\frac{(p')^2}{p^2} \leq 0 \qquad \text{for} \quad x \in (\varepsilon_1, 2]\,,$$

where $\varepsilon_1 \in [0,2)$ is a constant which will be determined later. This inequality is satisfied by the family (see appendix A.3)

$$p_k(x) = c_2\Big(c_1 + x^{1/k}\Big)^k\,, \qquad c_1 \in \mathbb{R}\,, \quad c_2 > 0\,, \quad k \in (0,2]\,. \tag{5.3.2}$$

In fact, for $k = 2$ the inequality is saturated. Assume $\operatorname{Tr}(s) \in (\varepsilon_1, 2]$. Then, given this particular p_k, setting $c_2 := 1$ and noting that the curvature term is non-negative, we obtain the differential inequality for the trace,

$$\left(\frac{\partial}{\partial t} - \Delta\right) \ln p_k(\operatorname{Tr}(s)) \geq 2\frac{p_k'}{p_k}\operatorname{Tr}(s)\|A\|^2 + \frac{1}{2}\|\nabla \ln p_k(\operatorname{Tr}(s))\|^2\,. \tag{5.3.3}$$

Lemma 5.3.2. *Assume (M, g_M) and (N, g_N) to be Riemann surfaces, M being compact and N complete. Denote by $\mathcal{C}_A$ the curvature terms in the evolution equation for $\|A\|^2$ (see lemma 3.2.4). Then, for any $\varepsilon, \delta > 0$ with $\delta \leq 3\varepsilon$, the second fundamental form satisfies the evolution inequality*

$$\left(\frac{\partial}{\partial t} - \Delta\right) \ln\Big(\delta t\|A\|^2 + \varepsilon\Big) \leq 3\|A\|^2 + \frac{1}{2}\|\nabla \ln(\delta t\|A\|^2 + \varepsilon)\|^2 + \frac{\delta t}{\delta t\|A\|^2 + \varepsilon}\mathcal{C}_A\,.$$

Proof. If M and N are flat, this is [STW14, Lemma 3.2]. Using the estimates from this lemma[1] and writing out the curvature terms, we obtain the evolution inequality

$$\left(\frac{\partial}{\partial t} - \Delta\right) \|A\|^2 \leq -2\|\nabla^\perp A\|^2 + 3\|A\|^4 + \mathcal{C}_A\,.$$

Then the same strategy as in the proof of corollary 5.2.3 yields

$$\left(\frac{\partial}{\partial t} - \Delta\right) \ln\Big(\delta t\|A\|^2 + \varepsilon\Big) \leq 3\|A\|^2 + \frac{1}{2}\|\nabla \ln(\delta t\|A\|^2 + \varepsilon)\|^2 + \frac{\delta t}{\delta t\|A\|^2 + \varepsilon}\mathcal{C}_A\,. \square$$

In the following statements, we calculate and estimate the curvature terms given by $\mathcal{C}_A$. Let us set

$$\mathcal{C}_1 := 4\sum_{i,j,k,l=1}^{2} (F^*\mathrm{R}_{M\times N})_{kilj}\Big(g_{M\times N}(A_{ij}, A_{kl}) - \delta^{kl}\sum_{p=1}^{2} g_{M\times N}(A_{ip}, A_{jp})\Big)\,,$$

[1] As pointed out in [STW14], one may apply [LL92, Theorem 1] to get the factor 3 for the norm of the second fundamental form.

$$\mathcal{C}_2 := 8 \sum_{i,k,l=1}^{2} \mathrm{R}_{M\times N}(A_{kl}, A_{ik}, \mathrm{d}F(e_l), \mathrm{d}F(e_i)),$$

$$\mathcal{C}_3 := 2 \sum_{i,k,l=1}^{2} \mathrm{R}_{M\times N}(A_{kl}, \mathrm{d}F(e_i), A_{kl}, \mathrm{d}F(e_i)),$$

$$\mathcal{C}_4 := 2 \sum_{i,k,l=1}^{2} (\nabla_{\mathrm{d}F(e_i)}\mathrm{R}_{M\times N})(A_{kl}, \mathrm{d}F(e_l), \mathrm{d}F(e_k), \mathrm{d}F(e_i)),$$

$$\mathcal{C}_5 := 2 \sum_{i,k,l=1}^{2} (\nabla_{\mathrm{d}F(e_k)}\mathrm{R}_{M\times N})(A_{kl}, \mathrm{d}F(e_i), \mathrm{d}F(e_l), \mathrm{d}F(e_i)),$$

so that $\mathcal{C}_A = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 + \mathcal{C}_5$.

Lemma 5.3.3. *The curvature term $\mathcal{C}_1$ is given by*

$$\mathcal{C}_1 = 4\frac{\sec_M + \lambda_1^2\lambda_2^2 \sec_N}{(1+\lambda_1^2)(1+\lambda_2^2)}\Big(2\,\mathrm{g}_{M\times N}(A_{11}, A_{22}) - 2\|A_{12}\|^2 - \|A\|^2\Big).$$

Proof. By the symmetries of the curvature tensor, only the index combinations

$$(k,i,l,j) \in \{(1,2,1,2),(1,2,2,1),(2,1,1,2),(2,1,2,1)\}$$

contribute to the sum. Therefore, we obtain

$$\begin{aligned}
&4 \sum_{i,j,k,l=1}^{2} (F^*\mathrm{R}_{M\times N})_{kilj}\mathrm{g}_{M\times N}(A_{ij}, A_{kl}) \\
&\quad = \quad 4(F^*\mathrm{R}_{M\times N})_{1212}\mathrm{g}_{M\times N}(A_{22}, A_{11}) + 4(F^*\mathrm{R}_{M\times N})_{1221}\mathrm{g}_{M\times N}(A_{21}, A_{12}) \\
&\qquad + 4(F^*\mathrm{R}_{M\times N})_{2112}\mathrm{g}_{M\times N}(A_{12}, A_{21}) + 4(F^*\mathrm{R}_{M\times N})_{2121}\mathrm{g}_{M\times N}(A_{11}, A_{22}) \\
&\quad = 8(F^*\mathrm{R}_{M\times N})_{1212}\Big(\mathrm{g}_{M\times N}(A_{11}, A_{22}) - \mathrm{g}_{M\times N}(A_{12}, A_{12})\Big)
\end{aligned}$$

and

$$\begin{aligned}
&-4 \sum_{i,j,k,p=1}^{2} (F^*\mathrm{R}_{M\times N})_{kikj}\mathrm{g}_{M\times N}(A_{ip}, A_{jp}) \\
&\quad = -4(F^*\mathrm{R}_{M\times N})_{1212} \sum_{p=1}^{2} \mathrm{g}_{M\times N}(A_{2p}, A_{2p}) \\
&\qquad\qquad - 4(F^*\mathrm{R}_{M\times N})_{2121} \sum_{p=1}^{2} \mathrm{g}_{M\times N}(A_{1p}, A_{1p}) \\
&\quad = -4(F^*\mathrm{R}_{M\times N})_{1212}\Big(2\mathrm{g}_{M\times N}(A_{12}, A_{12}) \\
&\qquad\qquad\qquad + \mathrm{g}_{M\times N}(A_{11}, A_{11}) + \mathrm{g}_{M\times N}(A_{22}, A_{22})\Big)
\end{aligned}$$

$$= -4(F^*\mathrm{R}_{M\times N})_{1212}\|A\|^2$$

The claim follows from

$$(F^*\mathrm{R}_{M\times N})_{1212} = \frac{1}{(1+\lambda_1^2)(1+\lambda_2^2)}(\sec_M + \lambda_1^2\lambda_2^2\sec_N)\,. \qquad \square$$

Lemma 5.3.4. *The curvature term $\mathcal{C}_2$ is given by*

$$\mathcal{C}_2 = 16\frac{\lambda_1\lambda_2(\sec_M+\sec_N)}{(1+\lambda_1^2)(1+\lambda_2^2)}\Big(A^1_{11}A^2_{12} + A^1_{12}A^2_{22} - A^2_{11}A^1_{12} - A^2_{12}A^1_{22}\Big)\,.$$

Proof. Let us consider the curvature tensor for two normal and two tangent directions. We calculate

$$\begin{aligned}&\mathrm{R}_{M\times N}(\xi_i,\xi_j,\mathrm{d}F(e_k),\mathrm{d}F(e_l))\\ &= \frac{1}{\sqrt{(1+\lambda_i^2)(1+\lambda_j^2)(1+\lambda_k^2)(1+\lambda_l^2)}}\Big(\lambda_i\lambda_j(\mathrm{R}_M)_{ijkl} + \lambda_k\lambda_l(\mathrm{R}_N)_{ijkl}\Big)\,.\end{aligned}$$

Using the symmetries of the curvature tensor, we see that these terms vanish except for the index combinations

$$(i,j,k,l) \in \{(1,2,1,2),(1,2,2,1),(2,1,1,2),(2,1,2,1)\}\,.$$

Evaluating the curvature corresponding to the first combination yields

$$\mathrm{R}_{M\times N}(\xi_1,\xi_2,\mathrm{d}F(e_1),\mathrm{d}F(e_2)) = \frac{\lambda_1\lambda_2}{(1+\lambda_1^2)(1+\lambda_2^2)}(\sec_M+\sec_N)$$

and the others follow from the symmetries of the Riemannian curvature tensor. The term $\mathcal{C}_2$ then can be evaluated as

$$\begin{aligned}\mathcal{C}_2 &= 8\sum_{i,k,l=1}^{2}\mathrm{R}_{M\times N}(A_{kl},A_{ik},\mathrm{d}F(e_l),\mathrm{d}F(e_i))\\ &= 8\sum_{i,k,l=1}^{2}\sum_{p,q=1}^{2}A^p_{kl}A^q_{ik}\mathrm{R}_{M\times N}(\xi_p,\xi_q,\mathrm{d}F(e_l),\mathrm{d}F(e_i))\\ &= \quad 8\sum_{k=1}^{2}\mathrm{R}_{M\times N}(\xi_1,\xi_2,\mathrm{d}F(e_1),\mathrm{d}F(e_2))\Big(A^1_{k1}A^2_{2k} - A^2_{k1}A^1_{2k}\Big)\\ &\quad -8\sum_{k=1}^{2}\mathrm{R}_{M\times N}(\xi_1,\xi_2,\mathrm{d}F(e_1),\mathrm{d}F(e_2))\Big(A^1_{k2}A^2_{1k} - A^2_{k2}A^1_{1k}\Big)\\ &= 16\frac{\lambda_1\lambda_2(\sec_M+\sec_N)}{(1+\lambda_1^2)(1+\lambda_2^2)}\sum_{k=1}^{2}\Big(A^1_{k1}A^2_{2k} - A^2_{k1}A^1_{2k}\Big)\\ &= 16\frac{\lambda_1\lambda_2(\sec_M+\sec_N)}{(1+\lambda_1^2)(1+\lambda_2^2)}\Big(A^1_{11}A^2_{12} + A^1_{12}A^2_{22} - A^2_{11}A^1_{12} - A^2_{12}A^1_{22}\Big)\,. \qquad \square\end{aligned}$$

Lemma 5.3.5. *The curvature term $\mathcal{C}_3$ is given by*

$$\mathcal{C}_3 = \frac{2}{(1+\lambda_1^2)(1+\lambda_2^2)}\Big((\lambda_2^2 \sec_M + \lambda_1^2 \sec_N)\|A^2\|^2 + (\lambda_1^2 \sec_M + \lambda_2^2 \sec_N)\|A^1\|^2\Big)\,.$$

Proof. We proceed as in the proof of lemma 5.3.4. First, for the curvature tensor we calculate

$$\mathrm{R}_{M\times N}(\xi_k, \mathrm{d}F(e_i), \xi_l, \mathrm{d}F(e_i)) = \frac{\lambda_k\lambda_l(\mathrm{R}_M)_{kili} + \lambda_i^2(\mathrm{R}_N)_{kili}}{(1+\lambda_i^2)\sqrt{(1+\lambda_k^2)(1+\lambda_l^2)}}\,.$$

Exploiting the symmetries of the curvature tensor, the only possibly non-vanishing curvature terms are given by the index combinations

$$(k,l,i) \in \{(1,1,2),(2,2,1)\}\,,$$

with corresponding curvature terms

$$\begin{aligned}
\mathrm{R}_{M\times N}(\xi_1, \mathrm{d}F(e_2), \xi_1, \mathrm{d}F(e_2)) &= \frac{\lambda_1^2 \sec_M + \lambda_2^2 \sec_N}{(1+\lambda_1^2)(1+\lambda_2^2)}\,,\\
\mathrm{R}_{M\times N}(\xi_2, \mathrm{d}F(e_1), \xi_2, \mathrm{d}F(e_1)) &= \frac{\lambda_2^2 \sec_M + \lambda_1^2 \sec_N}{(1+\lambda_1^2)(1+\lambda_2^2)}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{C}_3 &= 2\sum_{i,k,l=1}^{2} \mathrm{R}_{M\times N}(A_{kl}, \mathrm{d}F(e_i), A_{kl}, \mathrm{d}F(e_i))\\
&= 2\sum_{i,k,l=1}^{2}\sum_{p,q=1}^{2} A_{kl}^p A_{kl}^q \mathrm{R}_{M\times N}(\xi_p, \mathrm{d}F(e_i), \xi_q, \mathrm{d}F(e_i))\\
&= 2\sum_{i=1}^{2}\sum_{p=1}^{2} \|A^p\|^2 \mathrm{R}_{M\times N}(\xi_p, \mathrm{d}F(e_i), \xi_p, \mathrm{d}F(e_i))\\
&= 2\left(\|A^1\|^2\frac{\lambda_1^2 \sec_M + \lambda_2^2 \sec_N}{(1+\lambda_1^2)(1+\lambda_2^2)} + \|A^2\|^2\frac{\lambda_2^2 \sec_M + \lambda_1^2 \sec_N}{(1+\lambda_1^2)(1+\lambda_2^2)}\right)\,.
\end{aligned}$$

□

Before we continue to do similar calculations for the terms $\mathcal{C}_4$ and $\mathcal{C}_5$, we estimate the terms $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$.

Corollary 5.3.6. *Assume (M, g_M) and (N, g_N) to be Riemann surfaces, M being compact and N complete. Assume that there are constants $\sigma, \kappa_N \geq 0$, such that*

$$\sec_N \leq \sigma \leq \sec_M\,, \qquad \kappa_N := \sup_{x\in N} |\sec_N(x)| < \infty\,.$$

holds. Denote by κ_M the maximum of the sectional curvatures of M,

$$\kappa_M := \max_{p \in M} \sec_M(p)\,.$$

If $f : M \to N$ is a weakly area-decreasing map, then the curvature terms $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ are estimated by

$$\begin{aligned}
\mathcal{C}_1 &\leq \frac{4(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}\|A\|^2\kappa_N\,,\\
\mathcal{C}_2 &\leq \frac{4(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}\|A\|^2(\kappa_M+\kappa_N)\,,\\
\mathcal{C}_3 &\leq \frac{2(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}\|A\|^2(\sigma+\kappa_M)\,.
\end{aligned}$$

Proof. To prove the first inequality, note that

$$2\mathrm{g}_{M\times N}(A_{11}, A_{22}) - 2\|A_{12}\|^2 - \|A\|^2 = -\|A_{11}-A_{22}\|^2 - 4\|A_{12}\|^2 \leq 0\,.$$

Since $\sec_M \geq 0$ by assumption, we obtain

$$\begin{aligned}
\mathcal{C}_1 &\leq 4\frac{\lambda_1^2\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}\Big(2\mathrm{g}_{M\times N}(A_{11}, A_{22}) - 2\|A_{12}\|^2 - \|A\|^2\Big)\sec_N\\
&\leq 8\frac{\lambda_1^2\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}\|A\|^2\kappa_N\,.
\end{aligned}$$

From the area-decreasing condition $\lambda_1^2\lambda_2^2 \leq 1$ we further obtain

$$\lambda_1^2\lambda_2^2 \leq |\lambda_1\lambda_2| \leq \frac{1}{2}(\lambda_1^2+\lambda_2^2)\,,$$

so that the estimate for $\mathcal{C}_1$ follows. The estimates for $\mathcal{C}_2$ and $\mathcal{C}_3$ are straightforward calculations. □

Corollary 5.3.7. *Under the same assumptions as in the last corollary, it is*

$$\frac{\delta t}{\delta t\|A\|^2+\varepsilon}\Big(\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3\Big) \leq \frac{2(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}\Big(\sigma+3\kappa_M+4\kappa_N\Big)\,.$$

Proof. This is a direct consequence of the previous corollary and

$$\frac{\delta t\|A\|^2}{\delta t\|A\|^2+\varepsilon} \leq 1\,.$$

□

Recall the construction of the bases $\{\alpha_1, \alpha_2\}$ at $p \in M$ and $\{\beta_1, \beta_2\}$ at $f(p) \in N$ from section 2.3. With this notion and $v \in T_pM$, $w \in T_{f(p)}N$, let us set

$$(\nabla_v \mathrm{R}_M)_{ijkl} := (\nabla_v \mathrm{R}_M)(\alpha_i, \alpha_j, \alpha_k, \alpha_l)$$

and

$$(\nabla_w \mathrm{R}_N)_{ijkl} := (\nabla_w \mathrm{R}_N)(\beta_i, \beta_j, \beta_k, \beta_l)\,.$$

Lemma 5.3.8. *The curvature term $\mathcal{C}_4$ satisfies*

$$\begin{aligned}
\mathcal{C}_4 = \frac{2}{(1+\lambda_1^2)(1+\lambda_2^2)} \Bigg\{ &(\nabla_{\alpha_1} \mathrm{R}_M)_{1212} \left(A^1_{22} \frac{\lambda_1}{\sqrt{1+\lambda_1^2}} - A^2_{21} \frac{\lambda_2}{\sqrt{1+\lambda_1^2}} \right) \\
&+ (\nabla_{\alpha_2} \mathrm{R}_M)_{1212} \left(-A^1_{12} \frac{\lambda_1}{\sqrt{1+\lambda_2^2}} + A^2_{11} \frac{\lambda_2}{\sqrt{1+\lambda_2^2}} \right) \\
&+ (\nabla_{\beta_1} \mathrm{R}_N)_{1212} \left(-A^1_{22} \frac{\lambda_1^2\lambda_2^2}{\sqrt{1+\lambda_1^2}} + A^2_{21} \frac{\lambda_1^3\lambda_2}{\sqrt{1+\lambda_1^2}} \right) \\
&+ (\nabla_{\beta_2} \mathrm{R}_N)_{1212} \left(A^1_{12} \frac{\lambda_1\lambda_2^3}{\sqrt{1+\lambda_2^2}} - A^2_{11} \frac{\lambda_1^2\lambda_2^2}{\sqrt{1+\lambda_2^2}} \right) \Bigg\}\,.
\end{aligned}$$

Proof. Using the definition of $\mathcal{C}_4$, we calculate

$$\begin{aligned}
\mathcal{C}_4 &= 2 \sum_{i,k,l=1}^{2} (\nabla_{\mathrm{d}F(e_i)} \mathrm{R}_{M\times N})(A_{kl}, \mathrm{d}F(e_l), \mathrm{d}F(e_k), \mathrm{d}F(e_i)) \\
&= 2 \sum_{\alpha,i,k,l=1}^{2} A^\alpha_{kl} (\nabla_{\mathrm{d}F(e_i)} \mathrm{R}_{M\times N})(\xi_\alpha, \mathrm{d}F(e_l), \mathrm{d}F(e_k), \mathrm{d}F(e_i))\,.
\end{aligned}$$

Using the symmetries of $\mathrm{R}_{M\times N}$, we see that only the index combinations

$$(\alpha, l, k, i) \in \{(1,2,1,2), (1,2,2,1), (2,1,1,2), (2,1,2,1)\}$$

contribute to the sum. Therefore, we obtain the expression

$$\begin{aligned}
\mathcal{C}_4 = 2\Big\{ \quad &A^1_{12} (\nabla_{\mathrm{d}F(e_2)} \mathrm{R}_{M\times N})(\xi_1, \mathrm{d}F(e_2), \mathrm{d}F(e_1), \mathrm{d}F(e_2)) \\
&+ A^1_{22} (\nabla_{\mathrm{d}F(e_1)} \mathrm{R}_{M\times N})(\xi_1, \mathrm{d}F(e_2), \mathrm{d}F(e_2), \mathrm{d}F(e_1)) \\
&+ A^2_{11} (\nabla_{\mathrm{d}F(e_2)} \mathrm{R}_{M\times N})(\xi_2, \mathrm{d}F(e_1), \mathrm{d}F(e_1), \mathrm{d}F(e_2)) \\
&+ A^2_{21} (\nabla_{\mathrm{d}F(e_1)} \mathrm{R}_{M\times N})(\xi_2, \mathrm{d}F(e_1), \mathrm{d}F(e_2), \mathrm{d}F(e_1)) \Big\}\,.
\end{aligned}$$

This may be evaluated in terms of the singular values λ_1, λ_2 of $\mathrm{d}f$,

$$
\begin{aligned}
\mathcal{C}_4 = \frac{2}{(1+\lambda_1^2)(1+\lambda_2^2)} \Bigg\{ & \frac{A^1_{12}}{\sqrt{1+\lambda_2^2}} \Big(-\lambda_1 (\nabla_{\alpha_2} \mathrm{R}_M)_{1212} + \lambda_2^3 \lambda_1 (\nabla_{\beta_2} \mathrm{R}_N)_{1212} \Big) \\
& + \frac{A^1_{22}}{\sqrt{1+\lambda_1^2}} \Big(-\lambda_1 (\nabla_{\alpha_1} \mathrm{R}_M)_{1221} + \lambda_1^2 \lambda_2^2 (\nabla_{\beta_1} \mathrm{R}_N)_{1221} \Big) \\
& + \frac{A^2_{11}}{\sqrt{1+\lambda_2^2}} \Big(-\lambda_2 (\nabla_{\alpha_2} \mathrm{R}_M)_{2112} + \lambda_2^2 \lambda_1^2 (\nabla_{\beta_2} \mathrm{R}_N)_{2112} \Big) \\
& + \frac{A^2_{21}}{\sqrt{1+\lambda_1^2}} \Big(-\lambda_2 (\nabla_{\alpha_1} \mathrm{R}_M)_{2121} + \lambda_1^3 \lambda_2 (\nabla_{\beta_1} \mathrm{R}_N)_{2121} \Big) \Bigg\} \\
= \frac{2}{(1+\lambda_1^2)(1+\lambda_2^2)} \Bigg\{ & (\nabla_{\alpha_1} \mathrm{R}_M)_{1212} \left(A^1_{22} \frac{\lambda_1}{\sqrt{1+\lambda_1^2}} - A^2_{21} \frac{\lambda_2}{\sqrt{1+\lambda_1^2}} \right) \\
& + (\nabla_{\alpha_2} \mathrm{R}_M)_{1212} \left(-A^1_{12} \frac{\lambda_1}{\sqrt{1+\lambda_2^2}} + A^2_{11} \frac{\lambda_2}{\sqrt{1+\lambda_2^2}} \right) \\
& + (\nabla_{\beta_1} \mathrm{R}_N)_{1212} \left(-A^1_{22} \frac{\lambda_1^2 \lambda_2^2}{\sqrt{1+\lambda_1^2}} + A^2_{21} \frac{\lambda_1^3 \lambda_2}{\sqrt{1+\lambda_1^2}} \right) \\
& + (\nabla_{\beta_2} \mathrm{R}_N)_{1212} \left(A^1_{12} \frac{\lambda_1 \lambda_2^3}{\sqrt{1+\lambda_2^2}} - A^2_{11} \frac{\lambda_1^2 \lambda_2^2}{\sqrt{1+\lambda_2^2}} \right) \Bigg\} .
\end{aligned}
$$

□

Lemma 5.3.9. *The curvature term $\mathcal{C}_5$ satisfies*

$$
\begin{aligned}
\mathcal{C}_5 = \frac{2}{(1+\lambda_1^2)(1+\lambda_2^2)} \Bigg\{ & (\nabla_{\alpha_1} \mathrm{R}_M)_{1212} \left(-A^2_{12} \frac{\lambda_2}{\sqrt{1+\lambda_1^2}} - A^1_{11} \frac{\lambda_1}{\sqrt{1+\lambda_1^2}} \right) \\
& + (\nabla_{\alpha_2} \mathrm{R}_M)_{1212} \left(-A^2_{22} \frac{\lambda_2}{\sqrt{1+\lambda_2^2}} - A^1_{21} \frac{\lambda_1}{\sqrt{1+\lambda_2^2}} \right) \\
& + (\nabla_{\beta_1} \mathrm{R}_N)_{1212} \left(A^2_{12} \frac{\lambda_1^3 \lambda_2}{\sqrt{1+\lambda_1^2}} + A^1_{11} \frac{\lambda_1^2 \lambda_2^2}{\sqrt{1+\lambda_1^2}} \right)
\end{aligned}
$$

$$+(\nabla_{\beta_2}\mathrm{R}_N)_{1212}\left(\quad A^2_{22}\frac{\lambda_1^2\lambda_2^2}{\sqrt{1+\lambda_2^2}}+A^1_{21}\frac{\lambda_1\lambda_2^3}{\sqrt{1+\lambda_2^2}}\right)\Bigg\}.$$

Proof. Using the definition of $\mathcal{C}_5$, we calculate

$$\begin{aligned}\mathcal{C}_5 &= 2\sum_{i,l,k=1}^{2}(\nabla_{\mathrm{d}F(e_k)}\mathrm{R}_{M\times N})(A_{kl},\mathrm{d}F(e_i),\mathrm{d}F(e_l),\mathrm{d}F(e_i))\\ &= 2\sum_{\alpha,i,l,k=1}^{2}A^{\alpha}_{kl}(\nabla_{\mathrm{d}F(e_k)}\mathrm{R}_{M\times N})(\xi_\alpha,\mathrm{d}F(e_i),\mathrm{d}F(e_l),\mathrm{d}F(e_i))\,.\end{aligned}$$

Again exploiting the symmetries of the Riemannian curvature tensor, we obtain that only the terms with the index combinations

$$(\alpha,i,l,k)\in\{(2,1,2,1),(2,1,2,2),(1,2,1,1),(1,2,1,2)\}$$

contribute to the sum. Therefore,

$$\begin{aligned}\mathcal{C}_5 = 2\Big\{\ & A^2_{12}(\nabla_{\mathrm{d}F(e_1)}\mathrm{R}_{M\times N})(\xi_2,\mathrm{d}F(e_1),\mathrm{d}F(e_2),\mathrm{d}F(e_1))\\ &+A^2_{22}(\nabla_{\mathrm{d}F(e_2)}\mathrm{R}_{M\times N})(\xi_2,\mathrm{d}F(e_1),\mathrm{d}F(e_2),\mathrm{d}F(e_1))\\ &+A^1_{11}(\nabla_{\mathrm{d}F(e_1)}\mathrm{R}_{M\times N})(\xi_1,\mathrm{d}F(e_2),\mathrm{d}F(e_1),\mathrm{d}F(e_2))\\ &+A^1_{21}(\nabla_{\mathrm{d}F(e_2)}\mathrm{R}_{M\times N})(\xi_1,\mathrm{d}F(e_2),\mathrm{d}F(e_1),\mathrm{d}F(e_2))\Big\}\\ =\frac{2}{(1+\lambda_1^2)(1+\lambda_2^2)}&\Bigg\{\frac{A^2_{12}}{\sqrt{1+\lambda_1^2}}\Big(-\lambda_2(\nabla_{\alpha_1}\mathrm{R}_M)_{2121}+\lambda_1^3\lambda_2(\nabla_{\beta_1}\mathrm{R}_N)_{2121}\Big)\\ &+\frac{A^2_{22}}{\sqrt{1+\lambda_2^2}}\Big(-\lambda_2(\nabla_{\alpha_2}\mathrm{R}_M)_{2121}+\lambda_1^2\lambda_2^2(\nabla_{\beta_2}\mathrm{R}_N)_{2121}\Big)\\ &+\frac{A^1_{11}}{\sqrt{1+\lambda_1^2}}\Big(-\lambda_1(\nabla_{\alpha_1}\mathrm{R}_M)_{1212}+\lambda_1^2\lambda_2^2(\nabla_{\beta_1}\mathrm{R}_N)_{1212}\Big)\\ &+\frac{A^1_{21}}{\sqrt{1+\lambda_2^2}}\Big(-\lambda_1(\nabla_{\alpha_2}\mathrm{R}_M)_{1212}+\lambda_1\lambda_2^3(\nabla_{\beta_2}\mathrm{R}_N)_{1212}\Big)\Bigg\}\\ =\frac{2}{(1+\lambda_1^2)(1+\lambda_2^2)}&\Bigg\{(\nabla_{\alpha_1}\mathrm{R}_M)_{1212}\left(-A^2_{12}\frac{\lambda_2}{\sqrt{1+\lambda_1^2}}-A^1_{11}\frac{\lambda_1}{\sqrt{1+\lambda_1^2}}\right)\\ &+(\nabla_{\alpha_2}\mathrm{R}_M)_{1212}\left(-A^2_{22}\frac{\lambda_2}{\sqrt{1+\lambda_2^2}}-A^1_{21}\frac{\lambda_1}{\sqrt{1+\lambda_2^2}}\right)\end{aligned}$$

$$
\begin{aligned}
&+ (\nabla_{\beta_1} \mathrm{R}_N)_{1212} \left(A^2_{12} \frac{\lambda_1^3 \lambda_2}{\sqrt{1+\lambda_1^2}} + A^1_{11} \frac{\lambda_1^2 \lambda_2^2}{\sqrt{1+\lambda_1^2}} \right) \\
&+ (\nabla_{\beta_2} \mathrm{R}_N)_{1212} \left(A^2_{22} \frac{\lambda_1^2 \lambda_2^2}{\sqrt{1+\lambda_2^2}} + A^1_{21} \frac{\lambda_1 \lambda_2^3}{\sqrt{1+\lambda_2^2}} \right) \Bigg\} .
\end{aligned}
$$

$\square$

The sum of the terms $\mathcal{C}_4$ and $\mathcal{C}_5$ may be written as

$$
\begin{aligned}
\frac{2}{(1+\lambda_1^2)(1+\lambda_2^2)} \Bigg\{ & \frac{(\nabla_{\alpha_1} \mathrm{R}_M)_{1212}}{\sqrt{1+\lambda_1^2}} \Big(A^1_{22} \lambda_1 - 2A^2_{12} \lambda_2 - A^1_{11} \lambda_1 \Big) \\
&+ \frac{(\nabla_{\alpha_2} \mathrm{R}_M)_{1212}}{\sqrt{1+\lambda_2^2}} \Big(A^2_{11} \lambda_2 - A^2_{22} \lambda_2 - 2A^1_{12} \lambda_1 \Big) \\
&+ \frac{(\nabla_{\beta_1} \mathrm{R}_N)_{1212}}{\sqrt{1+\lambda_1^2}} \Big(2A^2_{12} \lambda_1^3 \lambda_2 + A^1_{11} \lambda_1^2 \lambda_2^2 - A^1_{22} \lambda_1^2 \lambda_2^2 \Big) \\
&+ \frac{(\nabla_{\beta_2} \mathrm{R}_N)_{1212}}{\sqrt{1+\lambda_2^2}} \Big(A^2_{22} \lambda_1^2 \lambda_2^2 + 2A^1_{12} \lambda_1 \lambda_2^3 - A^2_{11} \lambda_1^2 \lambda_2^2 \Big) \Bigg\} .
\end{aligned}
$$

Lemma 5.3.10. *Define $\delta_M \geq 0$ by*

$$
\delta_M := \max_{p \in M} \| \nabla^{g_M} \mathrm{R}_M \| (p)
$$

and assume there exists $\delta_N \geq 0$ with

$$
\delta_N := \sup_{q \in N} \| \nabla^{g_N} \mathrm{R}_N \| (q) < \infty .
$$

If $f : M \to N$ is a weakly area-decreasing map, then the estimate

$$
\begin{aligned}
\mathcal{C}_4 + \mathcal{C}_5 \leq\ & 2 \sqrt{\frac{2(\lambda_1^2 + \lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}} \Big(3 + \|A\|^2 \Big) \delta_M \\
&+ \frac{2(\lambda_1^2 + \lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)} \Big(5 + \|A\|^2 \Big) \frac{\delta_N}{2}
\end{aligned}
$$

is satisfied.

Proof. Let us set

$$
\gamma_1 := \lambda_1 (A^1_{22} - A^1_{11}) - 2\lambda_2 A^2_{12} ,
$$

$$\begin{aligned}
\gamma_2 &:= \lambda_2(A^2_{11} - A^2_{22}) - 2\lambda_1 A^1_{12}\,,\\
\gamma_3 &:= \lambda_1^2\lambda_2^2(A^1_{11} - A^1_{22}) + 2\lambda_1^3\lambda_2 A^2_{12}\,,\\
\gamma_4 &:= \lambda_1^2\lambda_2^2(A^2_{22} - A^2_{11}) + 2\lambda_1\lambda_2^3 A^1_{12}\,,
\end{aligned}$$

so that

$$\begin{aligned}
&\mathcal{C}_4 + \mathcal{C}_5\\
&= \frac{2}{(1+\lambda_1^2)(1+\lambda_2^2)}\Bigg\{(\nabla_{\alpha_1}\mathrm{R}_M)_{1212}\frac{\gamma_1}{\sqrt{1+\lambda_1^2}} + (\nabla_{\alpha_2}\mathrm{R}_M)_{1212}\frac{\gamma_2}{\sqrt{1+\lambda_2^2}}\\
&\qquad + (\nabla_{\beta_1}\mathrm{R}_N)_{1212}\frac{\gamma_3}{\sqrt{1+\lambda_1^2}} + (\nabla_{\beta_2}\mathrm{R}_N)_{1212}\frac{\gamma_4}{\sqrt{1+\lambda_2^2}}\Bigg\}\,.
\end{aligned}$$

Since the derivatives of the curvatures are bounded by assumption, we may estimate

$$\begin{aligned}
\mathcal{C}_4 + \mathcal{C}_5 &\le \frac{2}{(1+\lambda_1^2)(1+\lambda_2^2)}\Bigg\{\delta_M\left(\frac{|\gamma_1|}{\sqrt{1+\lambda_1^2}} + \frac{|\gamma_2|}{\sqrt{1+\lambda_2^2}}\right)\\
&\qquad + \delta_N\left(\frac{|\gamma_3|}{\sqrt{1+\lambda_1^2}} + \frac{|\gamma_4|}{\sqrt{1+\lambda_2^2}}\right)\Bigg\}\\
&\le \frac{2}{(1+\lambda_1^2)(1+\lambda_2^2)}\Big(\delta_M\big(|\gamma_1| + |\gamma_2|\big) + \delta_N\big(|\gamma_3| + |\gamma_4|\big)\Big)\,.
\end{aligned}$$

Then, we may use the Cauchy-Schwarz inequality and the equivalence of norms in $\mathbb{R}^2$ to obtain

$$\begin{aligned}
|\gamma_1| + |\gamma_2| &\le |\lambda_1||A^1_{22} - A^1_{11}| + 2|\lambda_2 A^2_{12}| + |\lambda_2||A^2_{11} - A^2_{22}| + 2|\lambda_1 A^1_{12}|\\
&\le \big(|\lambda_1| + |\lambda_2|\big)\big(|A^1_{22} - A^1_{11}| + 2|A^2_{12}| + |A^2_{11} - A^2_{22}| + 2|A^1_{12}|\big)\\
&\le \frac{1}{2}\big(|\lambda_1| + |\lambda_2|\big)\Big(6 + (A^1_{22} - A^1_{11})^2 + 2(A^2_{12})^2\\
&\qquad + (A^2_{11} - A^2_{22})^2 + 2(A^1_{12})^2\Big)\\
&\le \big(|\lambda_1| + |\lambda_2|\big)\Big(3 + (A^1_{22})^2 + (A^1_{11})^2 + (A^2_{12})^2\\
&\qquad + (A^2_{11})^2 + (A^2_{22})^2 + (A^1_{12})^2\Big)\\
&\le \big(|\lambda_1| + |\lambda_2|\big)\big(3 + \|A\|^2\big)\\
&\le \sqrt{2(\lambda_1^2 + \lambda_2^2)}\big(3 + \|A\|^2\big)\,.
\end{aligned}$$

It follows that

$$\begin{aligned}\frac{2}{(1+\lambda_1^2)(1+\lambda_2^2)}\delta_M\Big(|\gamma_1|+|\gamma_2|\Big) &\le 2\delta_M\frac{\sqrt{2(\lambda_1^2+\lambda_2^2)}}{(1+\lambda_1^2)(1+\lambda_2^2)}\Big(3+\|A\|^2\Big)\\ &\le 2\delta_M\sqrt{\frac{2(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}}\Big(3+\|A\|^2\Big),\end{aligned}$$

which establishes the first summand of the claimed inequality. For the remaining terms, note that

$$|\gamma_3|+|\gamma_4| \le \lambda_1^2\lambda_2^2\Big(|A_{11}^1-A_{22}^1|+|A_{22}^2-A_{11}^2|\Big)+2|\lambda_1\lambda_2|\Big(\lambda_1^2|A_{12}^2|+\lambda_2^2|A_{12}^1|\Big).$$

Using that for weakly area-decreasing maps we have $|\lambda_1\lambda_2|\le 1$, we calculate

$$\lambda_1^2\lambda_2^2 \le |\lambda_1\lambda_2| \le \frac{1}{2}(\lambda_1^2+\lambda_2^2)$$

and further

$$\lambda_k^2 A_{12}^l \le \frac{1}{2}\lambda_k^2\Big(1+(A_{12}^l)^2\Big) \le \frac{1}{2}(\lambda_1^2+\lambda_2^2)\Big(1+(A_{12}^l)^2\Big),$$

so that we may estimate

$$\begin{aligned}|\gamma_3|+|\gamma_4| &\le \lambda_1^2\lambda_2^2\Big(|A_{11}^1-A_{22}^1|+|A_{22}^2-A_{11}^2|\Big)\\ &\quad+(\lambda_1^2+\lambda_2^2)\Big(2+(A_{12}^2)^2+(A_{12}^1)^2\Big)\\ &\le \frac{1}{4}(\lambda_1^2+\lambda_2^2)\Big(2+(A_{11}^1-A_{22}^1)^2+(A_{22}^2-A_{11}^2)^2\Big)\\ &\quad+(\lambda_1^2+\lambda_2^2)\Big(2+(A_{12}^2)^2+(A_{12}^1)^2\Big)\\ &\le \frac{1}{2}(\lambda_1^2+\lambda_2^2)\Big(1+(A_{11}^1)^2+(A_{22}^1)^2+(A_{22}^2)^2+(A_{11}^2)^2\Big)\\ &\quad+(\lambda_1^2+\lambda_2^2)\Big(2+(A_{12}^2)^2+(A_{12}^1)^2\Big)\\ &= \frac{1}{2}(\lambda_1^2+\lambda_2^2)\Big(5+\|A\|^2\Big).\end{aligned}$$

This implies

$$\frac{2}{(1+\lambda_1^2)(1+\lambda_2^2)}\delta_N\Big(|\gamma_3|+|\gamma_4|\Big) \le \frac{\delta_N}{2}\frac{2(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}\Big(5+\|A\|^2\Big),$$

which shows the remaining part of the claim. □

Corollary 5.3.7 and lemma 5.3.10 imply that the evolution inequality for the second fundamental form (as given in lemma 5.3.2) may be estimated by

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right) \ln\left(\delta t\|A\|^2 + \varepsilon\right) \\
&\quad \leq 3\|A\|^2 + \frac{1}{2}\|\nabla \ln(\delta t\|A\|^2 + \varepsilon)\|^2 \\
&\qquad + \frac{2(\lambda_1^2 + \lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}(\sigma + 3\kappa_M + 4\kappa_N) \\
&\qquad + 2\frac{\delta t(3 + \|A\|^2)}{\delta t\|A\|^2 + \varepsilon}\sqrt{\frac{2(\lambda_1^2 + \lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}}\delta_M \\
&\qquad + \frac{1}{2}\frac{\delta t(5 + \|A\|^2)}{\delta t\|A\|^2 + \varepsilon}\frac{2(\lambda_1^2 + \lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}\delta_N\,.
\end{aligned} \tag{5.3.4}$$

The following statements put the coefficients of δ_N and δ_M into a more usable form.

Lemma 5.3.11. *For $\varepsilon, \eta > 0$, let $f_{\varepsilon,\eta} : \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be given by*

$$f_{\varepsilon,\eta}(t,a) := \frac{t(\eta + a)}{ta + \varepsilon}\,.$$

Then

$$f_{\varepsilon,\eta}(t,a) \leq \begin{cases} 1\,, & 0 \leq t \leq \frac{\varepsilon}{\eta}\,, \\ \frac{t\eta}{\varepsilon}\,, & \frac{\varepsilon}{\eta} < t\,. \end{cases}$$

Proof. We fix $t_0 \in \mathbb{R}^{\geq 0}$ and consider $f_{\varepsilon,\eta}(t_0,\cdot) : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ as a function of one variable. To get information about the maximum of this function, let us first investigate its monotonicity behavior, as given by its derivative. We calculate

$$\frac{\partial}{\partial a} f_{\varepsilon,\eta}(t_0,a) = \frac{t_0(t_0 a + \varepsilon) - t_0^2(\eta + a)}{(t_0 a + \varepsilon)^2} = \frac{t_0(\varepsilon - t_0\eta)}{(t_0 a + \varepsilon)^2}\,.$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial a} f_{\varepsilon,\eta}(t_0,a) > 0 \qquad &\Leftrightarrow \qquad 0 < t_0 < \frac{\varepsilon}{\eta}\,, \\
\frac{\partial}{\partial a} f_{\varepsilon,\eta}(t_0,a) = 0 \qquad &\Leftrightarrow \qquad t_0 = 0 \quad \text{or} \quad t_0 = \frac{\varepsilon}{\eta}\,, \\
\frac{\partial}{\partial a} f_{\varepsilon,\eta}(t_0,a) < 0 \qquad &\Leftrightarrow \qquad \frac{\varepsilon}{\eta} < t_0\,.
\end{aligned}$$

By monotonicity, this implies that for any fixed t_0 the maximal value of $f_{\varepsilon,\eta}(t_0,\cdot)$ is attained at either $a = 0$ (for $\varepsilon/\eta < t_0$) or $a \to \infty$ (for $0 \le t_0 \le \varepsilon/\eta$). The corresponding values are given by

$$f_{\varepsilon,\eta}(t_0,0) = \frac{t_0\eta}{\varepsilon}\,, \quad \lim_{a\to\infty} f_{\varepsilon,\eta}(0,a) = 0 \quad \text{and} \quad \lim_{\substack{a\to\infty\\ 0<t_0\le\varepsilon/\eta}} f_{\varepsilon,\eta}(t_0,a) = 1\,. \qquad \square$$

Let us consider the coefficient of δ_M in inequality (5.3.4). By using the estimate for the singular values in terms of det(s) (see inequality (5.1.6)) and lemma 5.3.11, we can estimate

$$\begin{aligned}
\frac{\delta t(3+\|A\|^2)}{\delta t\|A\|^2+\varepsilon}\sqrt{\frac{2(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}} &\le \frac{\delta t(3+\|A\|^2)}{\delta t\|A\|^2+\varepsilon}\sqrt{\frac{2c_1}{\exp(2\sigma t)+c_1}}\\
&\le \begin{cases} \sqrt{\frac{2c_1}{\exp(2\sigma t)+c_1}}\,, & 0\le t\le \frac{\varepsilon}{3\delta}\,,\\ \frac{3\delta t}{\varepsilon}\sqrt{\frac{2c_1}{\exp(2\sigma t)+c_1}}\,, & \frac{\varepsilon}{3\delta}<t\,,\end{cases}\\
&\le \begin{cases} \frac{\sqrt{2c_1}}{\exp(\sigma t)}\,, & 0\le t\le \frac{\varepsilon}{3\delta}\,,\\ \frac{3\delta t}{\varepsilon}\frac{\sqrt{2c_1}}{\exp(\sigma t)}\,, & \frac{\varepsilon}{3\delta}<t\,.\end{cases}
\end{aligned}$$

To estimate this further, note that $t\exp(-\sigma t/2)$ has its maximum at $t = 2/\sigma$, which implies

$$t\exp\left(-\frac{\sigma}{2}t\right) \le \frac{2}{\sigma}\exp(-1) = \frac{2}{\sigma\mathrm{e}}\,.$$

The coefficient of δ_M may therefore be estimated by

$$\frac{\delta t(3+\|A\|^2)}{\delta t\|A\|^2+\varepsilon}\sqrt{\frac{2(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}} \le \begin{cases} \sqrt{2c_1}\exp\left(-\frac{\sigma}{2}t\right)\,, & 0\le t\le\frac{\varepsilon}{3\delta}\,,\\ \frac{3\delta}{\varepsilon}\frac{2}{\sigma\mathrm{e}}\sqrt{2c_1}\exp\left(-\frac{\sigma}{2}t\right)\,, & \frac{\varepsilon}{3\delta}<t\,.\end{cases}$$

Let us set

$$\eta_1 := \max\left\{1, \frac{\delta}{\varepsilon}\frac{6}{\sigma\mathrm{e}}\right\}\,.$$

Then

$$\frac{\delta t(3+\|A\|^2)}{\delta t\|A\|^2+\varepsilon}\sqrt{\frac{2(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}} \le \eta_1\sqrt{2c_1}\exp\left(-\frac{\sigma}{2}t\right)\,. \tag{5.3.5}$$

Let us go on by estimating the coefficient of δ_N in inequality (5.3.4). In the same way as above, we obtain

$$\frac{\delta t(5+\|A\|^2)}{\delta t\|A\|^2+\varepsilon}\frac{2(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)} \le \frac{\delta t(5+\|A\|^2)}{\delta t\|A\|^2+\varepsilon}\frac{2c_1}{\exp(2\sigma t)+c_1}$$
$$\le \begin{cases} \frac{2c_1}{\exp(2\sigma t)+c_1}\,, & 0 \le t \le \frac{\varepsilon}{5\delta}\,,\\ \frac{5\delta t}{\varepsilon}\frac{2c_1}{\exp(2\sigma t)+c_1}\,, & \frac{\varepsilon}{5\delta} < t\,, \end{cases}$$
$$\le \begin{cases} \frac{2c_1}{\exp(\sigma t)}\,, & 0 \le t \le \frac{\varepsilon}{5\delta}\,,\\ \frac{5\delta}{\varepsilon}\frac{t}{\exp(\sigma t)}\frac{2c_1}{\exp(\sigma t)}\,, & \frac{\varepsilon}{5\delta} < t\,. \end{cases}$$

Again as above, we estimate $t\exp(-\sigma t)$ by its maximal value,

$$t\exp(-\sigma t) \le \frac{1}{\sigma}\exp(-1) = \frac{1}{\sigma \mathrm{e}}\,,$$

so that

$$\frac{\delta t(5+\|A\|^2)}{\delta t\|A\|^2+\varepsilon}\frac{2(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)} \le \begin{cases} \frac{2c_1}{\exp(\sigma t)}\,, & 0 \le t \le \frac{\varepsilon}{5\delta}\,,\\ \frac{5\delta}{\varepsilon}\frac{1}{\sigma \mathrm{e}}\frac{2c_1}{\exp(\sigma t)}\,, & \frac{\varepsilon}{5\delta} < t\,. \end{cases}$$

Setting

$$\eta_2 := \max\left\{1, \frac{\delta}{\varepsilon}\frac{5}{\sigma\mathrm{e}}\right\}\,,$$

we therefore have shown

$$\frac{\delta t(5+\|A\|^2)}{\delta t\|A\|^2+\varepsilon}\frac{2(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)} \le \eta_2\frac{2c_1}{\exp(\sigma t)}\,. \tag{5.3.6}$$

Inserting inequalities (5.3.5) and (5.3.6) into inequality (5.3.4), we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\ln(\delta t\|A\|^2+\varepsilon) \le 3\|A\|^2 &+ \frac{1}{2}\|\nabla\ln(\delta t\|A\|^2+\varepsilon)\|^2\\ &+\frac{2(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}(\sigma+3\kappa_M+4\kappa_N)\\ &+2\delta_M\eta_1\sqrt{2c_1}\exp\left(-\frac{\sigma}{2}t\right)\\ &+\delta_N\eta_2 c_1\exp(-\sigma t)\,. \end{aligned}\tag{5.3.7}$$

Theorem 5.3.12. *Let (M, g_M) and (N, g_N) to be Riemann surfaces, M being compact and N complete. Assume that there exists $\sigma > 0$, such that the sectional curvatures $\sec_M$ of M and $\sec_N$ of N satisfy the relation*

$$\sec_N \le \sigma \le \sec_M\,.$$

Further, assume that there exist $\kappa_N, \delta_N \geq 0$, such that

$$\kappa_N := \sup_{x \in N} |\sec_N(x)| < \infty\,, \qquad \|\nabla^{g_N} \mathrm{R}_N\| \leq \delta_N < \infty\,.$$

If $\mathrm{Tr}(\mathrm{s}) > \frac{2}{9}$ *is satisfied on* $M \times \{0\}$*, then under the mean curvature flow the estimate*

$$t\|A\|^2 \leq C$$

holds for some constant $C \geq 0$ *which only depends on* $\inf_{M \times \{0\}} \mathrm{Tr}(\mathrm{s})$ *and the curvature bounds* $\sigma, \kappa_M, \kappa_N$ *and* δ_M, δ_N*.*

Proof. Let $h_1, h_2, h_3 : \mathbb{R}^{\geq 0} \to \mathbb{R}^{>0}$ be smooth, positive functions, which will be chosen later. Using the evolution inequality (5.3.3) for $p_k(\mathrm{Tr}(\mathrm{s}))$, where we will fix $k \in (0, 2]$ later, together with inequality (5.3.7), we obtain

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right) \ln \frac{\delta t\|A\|^2 + \varepsilon}{h_1(t)h_2(t)h_3(t)p_k(\mathrm{Tr}(\mathrm{s}))} \\
&\leq \left(3 - 2\frac{p_k'}{p_k} \mathrm{Tr}(\mathrm{s})\right) \|A\|^2 \\
&\quad + \frac{1}{2} \left\langle \nabla \ln \frac{\delta t\|A\|^2 + \varepsilon}{h_1(t)h_2(t)h_3(t)p_k(\mathrm{Tr}(\mathrm{s}))}, \nabla \ln \left(\delta t\|A\|^2 + \varepsilon\right) p_k(\mathrm{Tr}(\mathrm{s})) \right\rangle \\
&\quad + \frac{2(\lambda_1^2 + \lambda_2^2)}{(1 + \lambda_1^2)(1 + \lambda_2^2)} (\sigma + 3\kappa_M + 4\kappa_N) - \frac{h_1'(t)}{h_1(t)} \\
&\quad + 2\delta_M \eta_1 \sqrt{2c_1} \exp\left(-\frac{\sigma}{2} t\right) - \frac{h_2'(t)}{h_2(t)} \\
&\quad + \delta_N \eta_2 c_1 \exp(-\sigma t) - \frac{h_3'(t)}{h_3(t)}\,.
\end{aligned}$$

Using the definition of p_k from equation (5.3.2), the coefficient of square norm of the second fundamental form is given by

$$3 - 2\frac{p_k'}{p_k} \mathrm{Tr}(\mathrm{s}) = 3 - 2\frac{(\mathrm{Tr}(\mathrm{s}))^{1/k}}{c_1 + (\mathrm{Tr}(\mathrm{s}))^{1/k}}\,,$$

which we wish to be negative. This leads to the constraints

$$c_1 < 0 \qquad \text{and} \qquad |c_1|^k < \mathrm{Tr}(\mathrm{s}) \leq 3^k |c_1|^k \qquad \text{for some} \quad k \in (0, 2]\,.$$

Using that under the assumptions of the theorem $\mathrm{Tr}(\mathrm{s})$ tends to 2 (see theorem 5.1.5), we set $c_1 := -\frac{1}{3} 2^{1/k}$, so that $|c_1|$ assumes its smallest value for $k = 2$. Therefore, we set $k := 2$, which means $|c_1|^2 = \frac{2}{9}$, and which is compatible with the condition $\mathrm{Tr}(\mathrm{s}) > \frac{2}{9} = |c_1|^2$. Setting $\delta := \varepsilon := 1$, we therefore have shown

$$\left(\frac{\partial}{\partial t} - \Delta\right) \ln \frac{t\|A\|^2 + 1}{h_1(t)h_2(t)h_3(t)p_2(\mathrm{Tr}(\mathrm{s}))}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left\langle \nabla \ln \frac{t\|A\|^2+1}{h_1(t)h_2(t)h_3(t)p_2(\mathrm{Tr(s)})}, \nabla \ln \left(t\|A\|^2+1\right)p_2(\mathrm{Tr(s)})\right\rangle \\
&\quad + \frac{2(\lambda_1^2+\lambda_2^2)}{(1+\lambda_1^2)(1+\lambda_2^2)}(\sigma+3\kappa_M+4\kappa_N) - \frac{h_1'(t)}{h_1(t)} \\
&\quad + 2\delta_M\eta_1\sqrt{2c_1}\exp\left(-\frac{\sigma}{2}t\right) - \frac{h_2'(t)}{h_2(t)} \\
&\quad + \delta_N\eta_2 c_1\exp(-\sigma t) - \frac{h_3'(t)}{h_3(t)}\,. \qquad (5.3.8)
\end{aligned}
$$

Using the same strategy as in the proof of theorem 5.2.4, we see that by setting

$$
h_1(t) := d_1\frac{\exp(2(\sigma+3\kappa_M+4\kappa_N)t)}{(c_1+\exp(2\sigma t))^{(\sigma+3\kappa_M+4\kappa_N)/\sigma}}\,, \qquad d_1>0\,,
$$

the third line in inequality (5.3.8) is non-positive. Further, setting

$$
\begin{aligned}
h_2(t) &:= d_2\exp\left\{-\frac{4\delta_M\eta_1}{\sigma}\sqrt{2c_1}\exp\left(-\frac{\sigma}{2}t\right)\right\}\,, \qquad d_2>0\,,\\
h_3(t) &:= d_3\exp\left\{-\frac{\delta_N\eta_2}{\sigma}c_1\exp(-\sigma t)\right\}\,, \qquad d_3>0\,,
\end{aligned}
$$

the last two lines in inequality (5.3.8) vanish identically. Explicitly, with these choices for h_1, h_2 and h_3, we have shown that

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)\ln\frac{t\|A\|^2+1}{h_1(t)h_2(t)h_3(t)p_2(\mathrm{Tr(s)})}\\
&\quad\leq \frac{1}{2}\left\langle \nabla \ln \frac{t\|A\|^2+1}{h_1(t)h_2(t)h_3(t)p_2(\mathrm{Tr(s)})}, \nabla \ln \left(t\|A\|^2+1\right)p_2(\mathrm{Tr(s)})\right\rangle\,.
\end{aligned}
$$

Now we may apply the maximum principle (theorem 2.2.3), by which we obtain

$$
\frac{t\|A\|^2+1}{h_1(t)h_2(t)h_3(t)p_2(\mathrm{Tr(s)})} \leq \frac{1}{h_1(0)h_2(0)h_3(0)}\sup_{M\times\{0\}}\frac{1}{p_2(\mathrm{Tr(s)})} =: \widetilde{C}\,.
$$

Note that by their definitions and since $d_1, d_2, d_3 > 0$, the functions h_1, h_2 and h_3 are monotonically increasing in t and

$$
\lim_{t\to\infty} h_1(t) = d_1\,, \qquad \lim_{t\to\infty} h_2(t) = d_2\,, \qquad \lim_{t\to\infty} h_3(t) = d_3\,.
$$

Since we are free to choose $d_1, d_2, d_3 > 0$, we set $d_1 := d_2 := d_3 := 1$. We calculate

$$
p_2'(x) = \left(c_1 + x^{1/2}\right)^1 x^{-1/2} > 0 \qquad \text{for} \qquad x\in(2/9,2]\,,
$$

which means p_2 is monotonically increasing on $(2/9, 2]$. This implies

$$
t\|A\|^2+1 \leq h_1(t)h_2(t)h_3(t)p_2(\mathrm{Tr(s)})\widetilde{C} \leq p_2(2)\widetilde{C}\,.
$$

Setting $C := p_2(2)\widetilde{C} - 1 \geq 0$ shows the theorem. □

As a corollary, we get the following result, which (for compact N) also follows from [LL11, Theorem 2] or [SS14c, Theorem A].

Corollary 5.3.13. *Under the assumptions of theorem 5.3.12, $\Gamma(f)$ converges to a totally geodesic submanifold of $M \times N$ under the mean curvature flow, which is the graph of a constant map.*

Proof. By theorem 5.3.12, the second fundamental tensor cannot blow up, so that we have a long-time solution. In particular, we also obtain $\lim_{t\to\infty} \|A\|^2 = 0$, so that for $t \to \infty$ the graph of f_t is totally geodesic. Further, the estimates in theorem 5.1.5 imply $\lim_{t\to\infty} \mathrm{Tr}(\mathrm{s}) = 2$, which is equivalent to $\lambda_1^2 \to 0$ and $\lambda_2^2 \to 0$ for $t \to \infty$. Therefore, the limiting map $\lim_{t\to\infty} f_t$ has to be the constant map. $\square$

Remark 5.3.14. Using $\|\vec{H}\|^2 \leq 2\|A\|^2$, theorem 5.3.12 also provides another proof of the decay estimate for $\|\vec{H}\|^2$ (which was shown in theorem 5.2.4) under stronger assumptions on the initial map and for surfaces N with bounded curvature $\sec_N$ and bounded curvature derivatives $\nabla^{g_N} \mathrm{R}_N$.

Chapter 6
Mean Curvature Flow of Entire Lipschitz Graphs

We consider the mean curvature flow of graphs in a non-compact setting. More precisely, we examine the flow of strictly length-decreasing maps $f : \mathbb{R}^m \to \mathbb{R}^n$. For this, we analyze the restriction of the tensor $\mathrm{s}_{M\times N}$ to the normal bundle. The case of Lagrangian graphs was treated by Chau, Chen and He [CCH12].

6.1 Evolution of Tensors in the Normal Bundle

Let us set up the geometric constructions involving the normal bundle as introduced in section 2.1 while adapting the notions to the graphic setting. A smooth map $F : M \to M \times N$ defines an orthogonal splitting (w. r. t. $\mathrm{g}_{M\times N}$) of the bundle

$$F^*T(M \times N) = \mathrm{d}F(TM) \oplus T^\perp M\,,$$

which induces a splitting of a vector field $v \in \Gamma(F^*T(M \times N))$ as

$$v = v^\top \oplus v^\perp\,.$$

The projection onto the normal part is denoted by $\mathrm{pr}^\perp : F^*T(M \times N) \to T^\perp M$ and the normal connection on $T^\perp M$ by $\nabla^\perp$. Let us also denote the evaluation of the second fundamental form in the direction of a vector $\xi \in \Gamma(T^\perp M)$ by

$$A_\xi(v, w) := \mathrm{g}_{M\times N}(A(v, w), \xi)\,.$$

In the same manner, we set

$$\vec{H}_\xi := \mathrm{g}_{M\times N}(\vec{H}, \xi)\,.$$

On $M \times N$, we are going to consider the tensor $\mathrm{s}_{M\times N}$ introduced in section 4.1 and its restrictions to the tangent and the normal bundle. The restriction of $\mathrm{s}_{M\times N}$ to the normal bundle is given by

$$\begin{aligned} \mathrm{s}^\perp &\in \mathrm{Sym}(F^*T^*(M\times N)\otimes F^*T^*(M\times N))\,,\\ \mathrm{s}^\perp(v,w) &:= \mathrm{s}_{M\times N}(\mathrm{pr}^\perp(v),\mathrm{pr}^\perp(w))\,. \end{aligned}$$

Note that if $\mathrm{s} \geq \varepsilon \mathrm{g}$ for some $\varepsilon > 0$ at a point $(x_0, t_0) \in M \times [0, T)$, it follows that $\mathrm{s}^\perp \leq -\varepsilon \mathrm{g}^\perp$ at $(F(x_0), t_0)$, where $\mathrm{g}^\perp$ denotes the restriction of the metric $\mathrm{g}_{M\times N}$ to the normal bundle.

In the sequel, we compare the eigenvalues of $\mathrm{s}^\perp$ with the eigenvalues of the following tensor constructed from the mean curvature vector,

$$\begin{aligned} \vartheta &\in \mathrm{Sym}(F^*T^*(M\times N)\otimes F^*T^*(M\times N))\,,\\ \vartheta(v,w) &:= \vec{H}_{\mathrm{pr}^\perp(v)}\vec{H}_{\mathrm{pr}^\perp(w)}\,. \end{aligned}$$

The tensors $\mathrm{s}^\perp$ and ϑ evolve under the mean curvature flow as given by the following statements.

Lemma 6.1.1 ([SS14b, Lemma 3.3]). *Let ξ be a unit vector normal to the evolving submanifold at a fixed point (x_0, t_0) in space-time. Then*

$$\begin{aligned} &(\nabla^\perp_{\partial_t}\mathrm{s}^\perp - \Delta^\perp \mathrm{s}^\perp)(\xi,\xi)\\ &\quad = 2\sum_{i,j=1}^m A_\xi(e_i,e_j)\mathrm{s}_{M\times N}(A(e_i,e_j),\xi)\\ &\qquad - 2\sum_{i,j,k=1}^m A_\xi(e_i,e_j)A_\xi(e_i,e_k)\mathrm{s}(e_j,e_k)\\ &\qquad - 2\sum_{i,j=1}^m \mathrm{R}_{M\times N}\Big(\mathrm{d}F(e_i),\mathrm{d}F(e_j),\mathrm{d}F(e_i),\xi\Big)\mathrm{s}_{M\times N}(\mathrm{d}F(e_j),\xi) \end{aligned}$$

for any g*-orthonormal basis $\{e_1, \ldots, e_m\}$ of $T_{x_0}M$.*

Lemma 6.1.2. *With respect to the normal connection $\nabla^\perp$, the mean curvature vector satisfies the following evolution equation:*

$$(\nabla^\perp_{\partial_t} - \Delta^\perp)\vec{H} = \sum_{i,j=1}^m A_{\vec{H}}(e_i,e_j)A(e_i,e_j) + \sum_{i=1}^m \mathrm{pr}^\perp\Big(\mathrm{R}_{M\times N}(\vec{H},\mathrm{d}F(e_i))\mathrm{d}F(e_i)\Big)\,.$$

Proof. We calculate the Laplacian of $\vec{H}$ with respect to the normal connection $\nabla^\perp$. Let $\{e_1, \ldots, e_m\}$ denote a local frame of TM, orthonormal with respect to the induced metric g. By definition, the derivative of $\vec{H}$ with respect to $\nabla^\perp$ is given by

$$\nabla^\perp_{v_2}\vec{H} = \mathrm{pr}^\perp\left(\nabla_{v_2}\vec{H}\right) = \nabla_{v_2}\vec{H} - \sum_{j=1}^m \mathrm{g}_{M\times N}\left(\nabla_{v_2}\vec{H}, \mathrm{d}F(e_j)\right)\mathrm{d}F(e_j)$$
$$= \nabla_{v_2}\vec{H} + \sum_{j=1}^m \mathrm{g}_{M\times N}\left(\vec{H}, A(v_2,e_j)\right)\mathrm{d}F(e_j)\,.$$

In the same way, we calculate

$$(\nabla^\perp)^2_{v_1,v_2}\vec{H} = \mathrm{pr}^\perp\left(\nabla^2_{v_1,v_2}\vec{H}\right) + \sum_{j=1}^m A_{\vec{H}}(v_2,e_j)A(v_1,e_j)\,.$$

Taking the trace, we conclude that the Laplacian for $\vec{H}$ with respect to the normal connection is given by

$$\Delta^\perp\vec{H} = \mathrm{pr}^\perp\left(\Delta\vec{H}\right) + \sum_{i,j=1}^m A_{\vec{H}}(e_i,e_j)A(e_i,e_j)\,.$$

Then the claim follows from lemma 3.2.2. □

Corollary 6.1.3 ([SS14b, Lemma 3.4]). *The symmetric* 2*-tensor* ϑ *evolves under the mean curvature flow according to the formula*

$$(\nabla^\perp_{\partial_t}\vartheta - \Delta^\perp\vartheta)(\xi,\xi) = 2\sum_{i,j=1}^m A_{\vec{H}}(e_i,e_j)A_\xi(e_i,e_j)\vec{H}_\xi - 2\sum_{i=1}^m \mathrm{g}_{M\times N}\left(\nabla^\perp_{e_i}\vec{H},\xi\right)^2$$
$$- 2\sum_{i=1}^m \mathrm{R}_{M\times N}\left(\vec{H},\mathrm{d}F(e_i),\mathrm{d}F(e_i),\xi\right)\vec{H}_\xi$$

for any normal vector ξ *in the normal bundle of the submanifold.*

In the following, we will focus on $M := \mathbb{R}^m$ and $N := \mathbb{R}^n$ with their Euclidean metrics $\mathrm{g}_{\mathbb{R}^m}$ and $\mathrm{g}_{\mathbb{R}^n}$, respectively. We will also use the notation

$$\langle u,v\rangle := \mathrm{g}_{\mathbb{R}^m\times\mathbb{R}^n}(u,v) \qquad \text{with} \quad u,v\in\Gamma(T(\mathbb{R}^m\times\mathbb{R}^n))$$

for the product metric $\mathrm{g}_{\mathbb{R}^m\times\mathbb{R}^n}$ on $\mathbb{R}^m\times\mathbb{R}^n$. We will assume that for all $t\in[0,T)$ for some $T>0$, the family of immersions given by $F_t := F(\cdot,t) : \mathbb{R}^m\to\mathbb{R}^m\times\mathbb{R}^n$ evolves under the mean curvature flow (3.1.1), i. e.

$$\frac{\partial F}{\partial t}(x,t) = \vec{H}(x,t)\,, \tag{6.1.1}$$

is satisfied for all $t\in[0,T)$ and all $x\in\mathbb{R}^m$.

6.2 Preserved Quantities

In the case where this immersion has bounded geometry (a precise definition is given below), we show that the strictly length-decreasing condition is preserved under the mean curvature flow. Similar to the compact case, we obtain the result by analyzing the tensor s.

Let $F(x,t)$ be a smooth solution to the mean curvature flow (6.1.1) on $\mathbb{R}^m \times [0,T)$ for some $0 < T \le \infty$ with initial condition $F(x,0) = F_0(x)$ for all $x \in \mathbb{R}^m$. Since the ambient manifold is the Euclidean space $\mathbb{R}^m \times \mathbb{R}^n$, the second fundamental form A of the graph $\Gamma(f) \subset \mathbb{R}^m \times \mathbb{R}^n$ is given by

$$A(v,w) = \mathrm{D}^{\mathrm{g}_{\mathbb{R}^m\times\mathbb{R}^n}}_{\mathrm{d}F(v)}(\mathrm{d}F(w)) - \mathrm{d}F(\nabla_v w)\,,$$

where $\mathrm{D}^{\mathrm{g}_{\mathbb{R}^m\times\mathbb{R}^n}}$ is the usual derivative on $\mathbb{R}^m \times \mathbb{R}^n$ and ∇ denotes the Levi-Civita connection of the induced metric $\mathrm{g} = F^*\mathrm{g}_{\mathbb{R}^m\times\mathbb{R}^n}$.

Definition 6.2.1. Let $F(x,t)$ be a smooth solution to the mean curvature flow (6.1.1) on $\mathbb{R}^m \times [0,T)$ for some $0 < T \le \infty$, such that for each $t \in [0,T)$ and non-negative integer k, $F(\mathbb{R}^m, t) \subset \mathbb{R}^m \times \mathbb{R}^n$ satisfies

$$\sup_{x\in\mathbb{R}^m} \|\nabla^k A(x,t)\| < \infty\,, \tag{6.2.1}$$

$$C_1(t)\mathrm{g}_{\mathbb{R}^m} \le \mathrm{g} \le C_2(t)\mathrm{g}_{\mathbb{R}^m}\,, \tag{6.2.2}$$

where $C_1(t)$ and $C_2(t)$ for each $t \in [0,T)$ are finite, positive constants depending only on t, and $\mathrm{g}_{\mathbb{R}^m\times\mathbb{R}^n}$ and $\mathrm{g}_{\mathbb{R}^m}$ are the standard Euclidean metrics on $\mathbb{R}^m \times \mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then we will say that the family of embeddings $F : \mathbb{R}^m \times [0,T) \to \mathbb{R}^m \times \mathbb{R}^n$ has *bounded geometry*.

Let $F_0(x) := (x, f_0(x))$. If the map $f_0 : \mathbb{R}^m \to \mathbb{R}^n$ is strictly length-decreasing such that $f_0^*\mathrm{g}_{\mathbb{R}^n} \le (1-\delta)\mathrm{g}_{\mathbb{R}^m}$ for some constant $\delta \in (0,1]$, we will show that this property is preserved under the mean curvature flow. In the case where domain and target are the same, i. e. $f_0 : \mathbb{R}^n \to \mathbb{R}^n$, and the embedding $F_0(x)$ is Lagrange, this behavior was observed in [CCH12, Lemma 3.1].

Following [CCH12], we consider a tensor which is derived from $\mathrm{s} - \varepsilon\mathrm{g}$ by multiplying s with a function which grows for $\|x\| \to \infty$ and then by taking an appropriate limit. For $R > 0$, let

$$\phi_R(x) := 1 + \frac{\|x\|^2_{\mathbb{R}^m}}{R^2}\,, \tag{6.2.3}$$

where $\|\cdot\|_{\mathbb{R}^m}$ is the Euclidean norm on $\mathbb{R}^m$. Further, for $R, \sigma > 0$ and $0 < \mu < \varepsilon$, set

$$\psi_{|(x,t)} := \mathrm{e}^{\sigma t}\phi_R(x)\mathrm{s}_{|(x,t)} - (\varepsilon - \mu)\mathrm{g}_{|(x,t)}\,. \tag{6.2.4}$$

Lemma 6.2.2. *Let $F(x,t)$ be a smooth solution to* (6.1.1) *with bounded geometry for $t \in [0,T)$ and assume there exists $\varepsilon > 0$, such that $\mathrm{s} - \varepsilon \mathrm{g} \geq 0$. Fix any $T' \in [0,T)$ and let $v \in \Gamma(T\mathbb{R}^m)$ and $\xi \in \Gamma(T^\perp \mathbb{R}^m)$. Then at a fixed point $(x_0, t_0) \in \mathbb{R}^m \times [0,T']$, the following estimates hold,*

$$
\begin{aligned}
-c(T')\frac{\|x_0\|_{\mathbb{R}^m}}{R^2}\mathrm{s}(v,v) \leq \langle \nabla\phi_R, (\nabla \mathrm{s})(v,v)\rangle \leq c(T')\frac{\|x_0\|_{\mathbb{R}^m}}{R^2}\mathrm{s}(v,v)\,,\\
c(T')\frac{\|x_0\|_{\mathbb{R}^m}}{R^2}\mathrm{s}(\xi,\xi) \leq \langle \nabla\phi_R, (\nabla^\perp \mathrm{s}^\perp)(\xi,\xi)\rangle \leq -c(T')\frac{\|x_0\|_{\mathbb{R}^m}}{R^2}\mathrm{s}^\perp(\xi,\xi)\,,\\
|\,\Delta\phi_R| \leq c(T')\left(\frac{1}{R^2} + \frac{\|x_0\|_{\mathbb{R}^m}}{R^2}\right)\,,
\end{aligned}
$$

where $c(T') \geq 0$ is a constant depending only on T'.

Proof. For ϕ_R, we calculate

$$
\begin{aligned}
\mathrm{d}\phi_R = \nabla\phi_R &= \sum_{i=1}^m \frac{2x_i}{R^2}\mathrm{d}x^i\,,\\
\Delta\phi_R &= \sum_{a,b=1}^m \mathrm{g}^{ab}(\nabla\mathrm{d}\phi_R)(\partial_{x^a},\partial_{x^b})\\
&= \frac{2}{R^2}\sum_{a,b=1}^m \mathrm{g}^{ab}\sum_{i,j=1}^m\left\{\delta_{ij}\mathrm{d}x^j\otimes\mathrm{d}x^i + x_i\sum_{k=1}^m \Gamma^i_{jk}\mathrm{d}x^j\otimes\mathrm{d}x^k\right\}(\partial_{x^a},\partial_{x^b})\\
&= \frac{2}{R^2}\sum_{a,b=1}^m \mathrm{g}^{ab}\left\{\delta_{ab} + \sum_{i=1}^m x_i\Gamma^i_{ab}\right\}\\
&= \frac{2}{R^2}\sum_{a=1}^m \mathrm{g}^{aa} + \frac{2}{R^2}\sum_{a,b,i=1}^m \mathrm{g}^{ab}\Gamma^i_{ab}x_i\\
&= \frac{2}{R^2}\sum_{a=1}^m \mathrm{g}^{aa} + \frac{2}{R^2}\frac{1}{\sqrt{\det(\mathrm{g})}}\sum_{i,j=1}^m \frac{\partial}{\partial x^j}\left\{\mathrm{g}^{ji}\sqrt{\det(\mathrm{g})}\right\}x_i\,.
\end{aligned}
$$

By the assumption (6.2.2), g is uniformly equivalent to $\mathrm{g}_{\mathbb{R}^m}$ on $\mathbb{R}^m \times [0,T']$ up to a constant depending only on T'. Further, by equation (6.2.1), its derivatives remain bounded on $\mathbb{R}^m \times [0,T']$. Together with equation (6.2.1), this implies

$$
|\,\Delta\phi_R| \leq c(T')\left(\frac{1}{R^2} + \frac{\|x_0\|_{\mathbb{R}^m}}{R^2}\right)\,.
$$

To estimate the derivatives, note that

$$
(\nabla_u \mathrm{s})(v,w) = \mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}(A(u,v),\mathrm{d}F(w)) + \mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}(\mathrm{d}F(v),A(u,w))\,.
$$

Therefore, the bounded geometry assumptions (6.2.1) and (6.2.2) imply that s and ∇s are uniformly bounded on $\mathbb{R}^m \times [0,T']$ by a constant $c(T')$ depending only on T'. Thus, also using the assumption $\varepsilon \mathrm{g} \le \mathrm{s}$, at (x_0, t_0) we have

$$-c(T')\frac{\|x_0\|_{\mathbb{R}^m}}{R^2}\mathrm{s}(v,v) \le \langle \nabla\phi_R, (\nabla \mathrm{s})(v,v)\rangle \le c(T')\frac{\|x_0\|_{\mathbb{R}^m}}{R^2}\mathrm{s}(v,v)\,.$$

The estimates for $\nabla^\perp \mathrm{s}^\perp$ follow from $\mathrm{s}^\perp \le -\varepsilon \mathrm{g}^\perp$ and

$$(\nabla^\perp_v \mathrm{s}^\perp)(\xi,\eta) = -\sum_{j=1}^m A_\xi(v,e_j)\mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}(\mathrm{d}F(e_j),\eta) - \sum_{j=1}^m A_\eta(v,e_j)\mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}(\mathrm{d}F(e_j),\xi)$$

(see [SS14b, Proof of Lemma 3.3]), together with the above arguments. □

Lemma 6.2.3. *Under the mean curvature flow, the tensor* ψ *evolves according to the equation*

$$\begin{aligned}(\nabla_{\partial_t}\psi - \Delta\psi)(u_1,u_2)&\\ = -\psi(\mathrm{Ric}\,u_1,u_2) &- \psi(u_1,\mathrm{Ric}\,u_2)\\ &+ 2(\varepsilon-\mu)\sum_{k=1}^m \langle A(u_1,e_k),A(u_2,e_k)\rangle\\ &- 2\mathrm{e}^{\sigma t}\phi_R\sum_{k=1}^m \mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}(A(u_1,e_k),A(u_2,e_k))\\ &- \mathrm{e}^{\sigma t}\Big\{(\Delta\phi_R)\mathrm{s}(u_1,u_2) + 2\langle\nabla\phi_R,(\nabla \mathrm{s})(u_1,u_2)\rangle - \sigma\phi_R \mathrm{s}(u_1,u_2)\Big\}\end{aligned}$$

for any $u_1, u_2 \in \Gamma(T\mathbb{R}^m)$.

Proof. We calculate

$$\begin{aligned}(\nabla_{\partial_t}\psi)(u_1,u_2) = \mathrm{e}^{\sigma t}\phi_R(\nabla_{\partial_t}\mathrm{s})(u_1,u_2) &- (\varepsilon-\mu)(\nabla_{\partial_t}\mathrm{g})(u_1,u_2)\\ &+ \sigma \mathrm{e}^{\sigma t}\phi_R \mathrm{s}(u_1,u_2)\,.\end{aligned}$$

Using the evolution equation for the tensor s (see lemma 4.2.1) and time evolution of the metric $(\nabla_{\partial_t}\mathrm{g})(u_1,u_2) = -2\langle A(u_1,u_2),\vec{H}\rangle$, we obtain

$$\begin{aligned}(\nabla_{\partial_t}\psi)(u_1,u_2) = \mathrm{e}^{\sigma t}\phi_R(\Delta \mathrm{s})(u_1,u_2) &- \mathrm{e}^{\sigma t}\phi_R\Big\{\mathrm{s}(\mathrm{Ric}\,u_1,u_2) + \mathrm{s}(u_1,\mathrm{Ric}\,u_2)\Big\}\\ &- 2\mathrm{e}^{\sigma t}\phi_R\sum_{k=1}^m \mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}(A(e_k,u_1),A(e_k,u_2))\\ &+ 2(\varepsilon-\mu)\langle A(u_1,u_2),\vec{H}\rangle + \sigma \mathrm{e}^{\sigma t}\phi_R \mathrm{s}(u_1,u_2)\,.\end{aligned}$$

On the other hand, we have

$$(\Delta\psi)(u_1,u_2) = \mathrm{e}^{\sigma t}(\Delta\phi_R)\mathrm{s}(u_1,u_2) + 2\mathrm{e}^{\sigma t}\langle\nabla\phi_R,(\nabla \mathrm{s})(u_1,u_2)\rangle + \mathrm{e}^{\sigma t}\phi_R(\Delta \mathrm{s})(u_1,u_2)$$

and, from the Gauß equation (2.1.1),

$$2\langle A(u_1,u_2),\vec{H}\rangle = \mathrm{g}(\mathrm{Ric}\,u_1,u_2)+\mathrm{g}(u_1,\mathrm{Ric}\,u_2)+2\sum_{k=1}^{m}\langle A(e_k,u_1),A(e_k,u_2)\rangle\,,$$

so that the claim follows. □

Lemma 6.2.4. *Let $F(x,t)$ be a smooth solution to* (6.1.1) *with bounded geometry for $t\in[0,T)$. Assume there exists $\varepsilon>0$ with $\mathrm{s}-\varepsilon\mathrm{g}\geq 0$ at $t=0$, and $\mathrm{s}-\frac{\varepsilon}{2}\mathrm{g}\geq 0$ for all $t\in[0,T)$. Then it is $\mathrm{s}-\varepsilon\mathrm{g}\geq 0$ for all $t\in[0,T)$.*

Proof. We will first show the following. Fix any $T'\in[0,T)$, $\sigma>0$ and $\mu<\varepsilon$. Then there exists R_0 depending only on σ and T', such that $\psi>0$ on $\mathbb{R}^m\times[0,T')$ for all $R\geq R_0$.

We argue by contradiction. So suppose the claim is false, meaning ψ is not positive definite on $\mathbb{R}^m\times[0,T']$ for some $R\geq R_0$. Then as $\psi>0$ on $\mathbb{R}^m\times\{0\}$, $\mathrm{s}-\frac{\varepsilon}{2}\mathrm{g}\geq 0$ on $[0,T)$ and $\phi_R(x)\to\infty$ as $\|x\|\to\infty$, it follows that $\psi>0$ outside some compact set $K\subset\mathbb{R}^m$ for all $t\in[0,T)$. We conclude that there exists $(x_0,t_0)\in K\times[0,T']$ such that ψ has a zero eigenvalue at (x_0,t_0) and that t_0 is the first such time. In other words, we have $\psi_{|(x_0,t_0)}(v,w)=0$ for some nonzero vector v and all w, and $\psi>0$ on $\mathbb{R}^m\times[0,t_0)$. Now extend v locally as follows: at the time slice t_0, with respect to the pullback metric $\mathrm{g}_{|(x,t_0)}$, we parallel translate v along radial geodesics in a normal neighborhood[1] U around x_0, then set $v_{|(x,t)}=v_{|(x,t_0)}$ for $x\in U$ and $t\leq t_0$. According to the second derivative criterion (see lemma 2.2.5), at the point (x_0,t_0) we have

$$(\nabla_{\partial_t}\psi)(v,v)\leq 0 \qquad\text{and}\qquad (\Delta\psi)(v,v)\geq 0\,. \tag{6.2.5}$$

On the other hand, we estimate the terms in the evolution equation for ψ from lemma 6.2.3. First, consider the normal bundle terms. Let $\{\xi_1,\ldots,\xi_n\}$ denote the normal basis constructed in section 2.3. By assumption $\mathrm{s}\geq\frac{\varepsilon}{2}\mathrm{g}$ holds in $[0,T)$, so that we may use equation (4.1.2) to evaluate $\mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}$ on the normal bundle to obtain

$$\sum_{k=1}^{m}\mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}(A(u,e_k),A(u,e_k))=\sum_{k=1}^{m}\sum_{i=1}^{n}A^2_{\xi_i}(u,e_k)\mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}(\xi_i,\xi_i)\leq 0 \tag{6.2.6}$$

for any $t\in[0,T)$. Further, at the point (x_0,t_0) (noting $\psi_{|(x_0,t_0)}(v,v)=0$) we get

[1] U a neighborhood around x_0 with respect to normal coordinates, such that there is an open subset $V\subset T_{x_0}\mathbb{R}^m$ with $0\in V$ and $\exp_{x_0}$ is a diffeomorphism between V and U.

$$
\begin{aligned}
&(\nabla_{\partial_t}\psi)(v,v) - (\Delta\psi)(v,v) \\
&\quad = \quad 2(\varepsilon-\mu)\sum_{k=1}^{m}\langle A(v,e_k),A(v,e_k)\rangle \\
&\qquad\quad - 2\mathrm{e}^{\sigma t}\phi_R\sum_{k=1}^{m}\mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}(A(v,e_k),A(v,e_k)) \\
&\qquad\quad - \mathrm{e}^{\sigma t}\Big\{(\Delta\phi_R)\mathrm{s}(v,v) + 2\langle\nabla\phi_R,(\nabla\mathrm{s})(v,v)\rangle - \sigma\phi_R\mathrm{s}(v,v)\Big\} \\
&\quad \overset{\text{Lem. 6.2.2}}{\geq} \quad 2(\varepsilon-\mu)\sum_{k=1}^{m}\langle A(v,e_k),A(v,e_k)\rangle \\
&\qquad\quad - 2\mathrm{e}^{\sigma t}\phi_R\sum_{k=1}^{m}\mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}(A(v,e_k),A(v,e_k)) \\
&\qquad\quad - \mathrm{e}^{\sigma t}\left\{c(T')\left(\frac{1}{R^2}+\frac{\|x_0\|_{\mathbb{R}^m}}{R^2}\right) + 2c(T')\frac{\|x_0\|_{\mathbb{R}^m}}{R^2}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. -\sigma - \sigma\frac{\|x_0\|^2_{\mathbb{R}^m}}{R^2}\right\}\mathrm{s}(v,v) \\
&\quad \overset{\text{Eq. (6.2.6)}}{\geq} \quad 2(\varepsilon-\mu)\sum_{k=1}^{m}\langle A(v,e_k),A(v,e_k)\rangle \\
&\qquad\quad + \mathrm{e}^{\sigma t}\left\{\sigma + \sigma\frac{\|x_0\|^2_{\mathbb{R}^m}}{R^2} - 3c(T')\frac{\|x_0\|_{\mathbb{R}^m}}{R^2} - \frac{c(T')}{R^2}\right\}\mathrm{s}(v,v)\,.
\end{aligned}
$$

Note that by choosing $R_0 > 0$ (depending on σ and T') large enough, the term

$$
\frac{\sigma}{2} + \sigma\frac{\|x_0\|^2_{\mathbb{R}^m}}{R^2} - 3c(T')\frac{\|x_0\|_{\mathbb{R}^m}}{R^2} - \frac{c(T')}{R^2}
$$

is strictly positive for any $R \geq R_0$ and any $\|x_0\|_{\mathbb{R}^m}$. Continuing with the above calculation, we obtain

$$
\begin{aligned}
&(\nabla_{\partial_t}\psi)(v,v) - (\Delta\psi)(v,v) \\
&\quad > 2(\varepsilon-\mu)\sum_{k=1}^{m}\langle A(v,e_k),A(v,e_k)\rangle + \mathrm{e}^{\sigma t}\frac{\sigma}{2}\mathrm{s}(v,v) \\
&\quad > 0\,,
\end{aligned}
\tag{6.2.7}
$$

where in the last estimate we used $\varepsilon > \mu$ and $\mathrm{s} - \frac{\varepsilon}{2}\mathrm{g} \geq 0$. But then (6.2.7) contradicts (6.2.5), which shows the claim.

The statement of the lemma follows by first letting $R \to \infty$, then $\mu \to 0$, then $\sigma \to 0$ and finally $T' \to T$. □

We go on by removing the additional assumption from the previous lemma.

Lemma 6.2.5. *Let $F(x,t)$ be a smooth solution to* (6.1.1) *with bounded geometry for $t \in [0,T)$. If there exists $\varepsilon > 0$ with $\mathrm{s} - \varepsilon \mathrm{g} \geq 0$ at $t = 0$, then $\mathrm{s} - \varepsilon \mathrm{g} \geq 0$ for all $t \in [0,T)$.*

Proof. The proof is the same as [CCH12, Lemma 3.1, Step 2]. By lemma 6.2.4, we only need to remove the assumption $\mathrm{s} - \frac{\varepsilon}{2}\mathrm{g} \geq 0$ in $[0,T)$. First note that at $t = 0$ it holds

$$\mathrm{s}(v,v) - \frac{\varepsilon}{2}\mathrm{g}(v,v) \geq \varepsilon \mathrm{g}(v,v) - \frac{\varepsilon}{2}\mathrm{g}(v,v) = \frac{\varepsilon}{2}\mathrm{g}(v,v) \geq c\varepsilon \mathrm{g}_{\mathbb{R}^m}(v,v)$$

for some constant $c > 0$. Also, by the bounded geometry assumption on $F(x,t)$, the right hand side of the evolution equation of s is bounded (see lemma 4.2.1). Further, since $(\nabla_{\partial_t}\mathrm{g})(u,v) = -2\langle A(u,v), \vec{H}\rangle$, the same is true for the induced metric, so that for any $t \in [0,T)$ it is

$$\|\nabla_{\partial_t}\mathrm{s}\|\,,\ \|\nabla_{\partial_t}\mathrm{g}\| \leq C(t)\,,$$

where $C(t)$ is a constant depending only on t. Since $\mathrm{s} - \varepsilon\mathrm{g} \geq 0$ at $t = 0$, it follows that there is a maximal time $T_0 > 0$ such that $\mathrm{s} - \frac{\varepsilon}{2}\mathrm{g} > 0$ holds in $[0, T_0)$. From lemma 6.2.4 we know that $\mathrm{s} - \varepsilon\mathrm{g} \geq 0$ on $\mathbb{R}^m \times [0, T_0)$. If $T_0 \neq T$, by continuity, we also know that $\mathrm{s} - \varepsilon\mathrm{g} \geq 0$ on $\mathbb{R}^m \times [0, T_0]$. By the same argument for finding T_0 above, we can find some positive T_0' such that $\mathrm{s} - \frac{\varepsilon}{2}\mathrm{g} \geq 0$ holds in $\mathbb{R}^m \times [T_0, T_0 + T_0')$, where $[T_0, T_0 + T_0') \subset [T_0, T)$. But this contradicts the choice of T_0, so that $T_0 = T$. □

Lemma 6.2.6. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth length-decreasing map and evolve it by the mean curvature flow. Then each $F_t(\mathbb{R}^m)$ is the graph of a length-decreasing map for $t \in [0,T)$.*

Proof. We consider the Jacobian of the projection $\pi_{\mathbb{R}^m} : F(\mathbb{R}^m) \to \mathbb{R}^m$. It may be expressed as

$$v := \star\Big[(\pi_{\mathbb{R}^m} \circ F)^* \Omega_M\Big]\,,$$

where $\star : \Omega^k(\mathbb{R}^m) \to \Omega^{m-k}(\mathbb{R}^m)$ is the Hodge star operator with respect to the induced metric g on the graph. At a point $(x_0, t_0) \in \mathbb{R}^m \times [0,T)$ in space-time we may rewrite v as

$$v(x_0,t_0) = \frac{1}{\sqrt{\prod_{k=1}^m (1 + \lambda_k^2)}}\,,$$

where $0 \leq \lambda_1 \leq \cdots \leq \lambda_m$ are the singular values of the differential $\mathrm{d}f$ of f at the point (x_0, t_0).

Denote by $[0,T)$ the interval of existence of the flow and let $[0,T_g) \subset [0,T)$ denote the time interval on which the flow remains graphic. We will show that $T_g = T$.

By lemma 3.1.2, the flow has short-time existence, so that $T > 0$. Let us assume that $T_g < T$. Since by lemma 6.2.5 the length-decreasing property is preserved as long as the flow remains graphic, there exists a fixed $\widetilde{\varepsilon} \in [0,1]$, such that

$$\lambda_k^2 \leq 1 - \widetilde{\varepsilon} \qquad \text{holds for all } k = 1, \ldots, m\,.$$

This implies

$$v(x_0, t_0) \geq \frac{1}{\sqrt{\prod_{k=1}^m (2 - \widetilde{\varepsilon})}} > 0\,.$$

By continuity, letting $t_0 \to T_g$, we see that

$$\lim_{t_0 \to T_g} v(x_0, t_0) > 0\,.$$

Since $F_{t_0}(\mathbb{R}^m)$ is a graph at (x_0, t_0) exactly if the Jacobian of the projection from $F_{t_0}(\mathbb{R}^m)$ to $\mathbb{R}^m$ is positive, we see that $F_{t_0}(\mathbb{R}^m)$ is graphic at T_g, contradicting the choice of T_g. Therefore, $T_g = T$. □

6.3 A Priori Estimates

We derive a priori estimates for the mean curvature vector and for the height functions of the graph. The estimates will follow by analyzing the tensor

$$\chi := -\mathrm{e}^{\sigma t} \phi_R \mathrm{s}^\perp - \varepsilon_2 t \vartheta\,.$$

Lemma 6.3.1. *For any unit vector ξ normal to the evolving submanifold at a fixed point (x_0, t_0) in space-time, the tensor χ satisfies the equation*

$$\begin{aligned}
&(\nabla^\perp_{\partial_t} \chi - \Delta^\perp \chi)(\xi, \xi) \\
&= \mathrm{e}^{\sigma t} \Big\{ -\sigma \phi_R \mathrm{s}^\perp(\xi,\xi) + (\Delta^\perp \phi_R) \mathrm{s}^\perp(\xi,\xi) + 2 \langle \nabla^\perp \phi_R, (\nabla^\perp \mathrm{s}^\perp)(\xi,\xi) \rangle \Big\} \\
&\quad - 2 \mathrm{e}^{\sigma t} \phi_R \left\{ \sum_{i,j=1}^m A_\xi(e_i, e_j) \mathrm{s}_{\mathbb{R}^m \times \mathbb{R}^n}(A(e_i,e_j), \xi) \right. \\
&\qquad\qquad \left. - \sum_{i,j,k=1}^m A_\xi(e_i,e_j) A_\xi(e_i,e_k) \mathrm{s}(e_j, e_k) \right\} \\
&\quad - 2 \varepsilon_2 t \left\{ \sum_{i,j=1}^m A_{\vec{H}}(e_i,e_j) A_\xi(e_i,e_j) \vec{H}_\xi - \sum_{i=1}^m \langle \nabla^\perp_{e_i} \vec{H}, \xi \rangle^2 \right\} - \varepsilon_2 \vartheta(\xi,\xi)\,.
\end{aligned}$$

under the mean curvature flow.

Proof. Using lemma 6.1.1 and corollary 6.1.3, for any unit normal vector ξ at a point (x_0, t_0) we calculate

$$\begin{aligned}
(\nabla^{\perp}_{\partial_t}\chi)(\xi,\xi) &= -\mathrm{e}^{\sigma t}\phi_R\Big\{\sigma \mathrm{s}^{\perp}(\xi,\xi) + (\nabla^{\perp}_{\partial_t}\mathrm{s}^{\perp})(\xi,\xi)\Big\} \\
&\qquad - \varepsilon_2\vartheta(\xi,\xi) - \varepsilon_2 t(\nabla^{\perp}_{\partial_t}\vartheta)(\xi,\xi) \\
&= -\mathrm{e}^{\sigma t}\phi_R\left\{\sigma \mathrm{s}^{\perp}(\xi,\xi) + (\Delta^{\perp}\,\mathrm{s}^{\perp})(\xi,\xi)\right. \\
&\qquad\qquad +2\sum_{i,j=1}^{m} A_{\xi}(e_i,e_j)\mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}(A(e_i,e_j),\xi) \\
&\qquad\qquad \left. -2\sum_{i,j,k=1}^{m} A_{\xi}(e_i,e_j)A_{\xi}(e_i,e_k)\mathrm{s}(e_j,e_k)\right\} \\
&\qquad - \varepsilon_2\vartheta(\xi,\xi) \\
&\qquad - \varepsilon_2 t\left\{(\Delta^{\perp}\,\vartheta)(\xi,\xi)\right. \\
&\qquad\qquad \left. +\,2\sum_{i,j=1}^{m} A_{\vec{H}}(e_i,e_j)A_{\xi}(e_i,e_j)\vec{H}_{\xi} \; -2\sum_{i=1}^{m}\langle\nabla^{\perp}_{e_i}\vec{H},\xi\rangle^2\right\} \\
&= (\Delta^{\perp}\,\chi)(\xi,\xi) + \mathrm{e}^{\sigma t}\Big\{-\sigma\phi_R\mathrm{s}^{\perp}(\xi,\xi) + (\Delta^{\perp}\,\phi_R)\mathrm{s}^{\perp}(\xi,\xi) \\
&\qquad\qquad + 2\langle\nabla^{\perp}\phi_R, (\nabla^{\perp}\mathrm{s}^{\perp})(\xi,\xi)\rangle\Big\} \\
&\qquad - 2\mathrm{e}^{\sigma t}\phi_R\left\{\sum_{i,j=1}^{m} A_{\xi}(e_i,e_j)\mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}(A(e_i,e_j),\xi)\right. \\
&\qquad\qquad \left. -\sum_{i,j,k=1}^{m} A_{\xi}(e_i,e_j)A_{\xi}(e_i,e_k)\mathrm{s}(e_j,e_k)\right\} \\
&\qquad - 2\varepsilon_2 t\left\{\sum_{i,j=1}^{m} A_{\vec{H}}(e_i,e_j)A_{\xi}(e_i,e_j)\vec{H}_{\xi} - \sum_{i=1}^{m}\langle\nabla^{\perp}_{e_i}\vec{H},\xi\rangle^2\right\} \\
&\qquad - \varepsilon_2\vartheta(\xi,\xi)\,.
\end{aligned}$$

□

Lemma 6.3.2. *Let $F(x,t)$ be a smooth solution to* (6.1.1) *with bounded geometry for every $t \in [0,T)$ and suppose $\mathrm{s} - \varepsilon_1\mathrm{g} \geq 0$ on $[0,T)$ for some $\varepsilon_1 > 0$. Then there exists a constant $\varepsilon_2 > 0$ depending only on ε_1 and the dimension $m = \dim\mathbb{R}^m$, such that*

$$\mathrm{s}^{\perp} + \varepsilon_2 t\vartheta \leq 0$$

on $\mathbb{R}^m \times [0,T)$.

Proof. The proof follows the strategy layed out in [CCH12, Lemma 3.2], but adapts it to the different geometric situation. Fix any $T' \in [0,T)$. We will first show that we can choose $R_0 > 0$, such that $\chi \geq 0$ on $\mathbb{R}^m \times [0,T')$ for all $R \geq R_0$.

Suppose χ is not positive definite on $\mathbb{R}^m \times [0, T']$ for some $R > R_0$. Then, as $\chi > 0$ on $\mathbb{R}^m \times \{0\}$, $\mathrm{s} - \varepsilon_1 \mathrm{g} \geq 0$ (and therefore $\mathrm{s}^\perp + \varepsilon_1 \mathrm{g}^\perp \leq 0$) in $[0, T)$, $\phi_R(x) \to \infty$ as $\|x\| \to \infty$ and by the bounded geometry condition (6.2.1), it follows that $\chi > 0$ outside some compact set $K \subset \mathbb{R}^m$ for all $t \in [0, T']$. We conclude that there exists $(x_0, t_0) \in K \times [0, T']$, such that χ has a zero eigenvalue at (x_0, t_0) and that t_0 is the first such time. In other words, we have $\chi_{|(x_0,t_0)}(\xi, \eta) = 0$ for some nonzero vector ξ and all vectors η, and $\chi > 0$ on $\mathbb{R}^m \times [0, t_0)$. Extend ξ locally in space and time as in the proof of lemma 6.2.4. By the second derivative criterion (see lemma 2.2.5), at (x_0, t_0) we have

$$\chi(\xi, \eta) = 0\,, \quad (\nabla^\perp \chi)(\xi, \xi) = 0\,, \quad (\nabla^\perp_{\partial_t} \chi)(\xi, \xi) \leq 0 \quad \text{and} \quad (\Delta^\perp \chi)(\xi, \xi) \geq 0 \tag{6.3.1}$$

for all vectors η. On the other hand, we may estimate the terms in the evolution equation for χ (see lemma 6.3.1). Let us abbreviate the terms in the equation by defining

$$\mathcal{A} := \mathrm{e}^{\sigma t} \left\{ (\Delta^\perp \phi_R) \mathrm{s}^\perp(\xi, \xi) + 2 \langle \nabla^\perp \phi_R, (\nabla^\perp \mathrm{s}^\perp)(\xi, \xi) \rangle - \sigma \phi_R \mathrm{s}^\perp(\xi, \xi) \right\}$$

and

$$\begin{aligned} \mathcal{B} := & -2\mathrm{e}^{\sigma t} \phi_R \sum_{i,j=1}^m A_\xi(e_i, e_j) \mathrm{s}_{\mathbb{R}^m \times \mathbb{R}^n}(A(e_i, e_j), \xi) \\ & + 2\mathrm{e}^{\sigma t} \phi_R \sum_{i,j,k=1}^m A_\xi(e_i, e_j) A_\xi(e_i, e_k) \mathrm{s}(e_j, e_k) \\ & - 2\varepsilon_2 t \left\{ \sum_{i,j=1}^m A_{\vec{H}}(e_i, e_j) A_\xi(e_i, e_j) \vec{H}_\xi - \sum_{i=1}^m \langle \nabla^\perp_{e_i} \vec{H}, \xi \rangle^2 \right\} - \varepsilon_2 \vec{H}^2_\xi \,. \end{aligned}$$

Then at (x_0, t_0) it is

$$0 \overset{\text{Eq. (6.3.1)}}{\geq} (\nabla^\perp_{\partial_t} \chi)(\xi, \xi) - (\Delta^\perp \chi)(\xi, \xi) = \mathcal{A} + \mathcal{B} \,. \tag{6.3.2}$$

From the assumption $\mathrm{s} - \varepsilon_1 \mathrm{g} \geq 0$ it follows that

$$\mathrm{s}^\perp(\xi, \xi) \leq -\varepsilon_1 \mathrm{g}_{\mathbb{R}^m \times \mathbb{R}^n}(\xi, \xi) < 0 \qquad \text{on} \qquad [0, T'] \,.$$

Using the estimates from lemma 6.2.2 and the definition of ϕ_R (see equation (6.2.3)), we calculate

$$
\begin{aligned}
\mathcal{A} \geq \mathrm{e}^{\sigma t_0} \Bigg\{ c(T') \left(\frac{1}{R^2} + \frac{\|x_0\|_{\mathbb{R}^m}}{R^2} \right) \mathrm{s}^{\perp}(\xi, \xi) + 2c(T') \frac{\|x_0\|_{\mathbb{R}^m}}{R^2} \mathrm{s}^{\perp}(\xi, \xi) \\
- \sigma \left(1 + \frac{\|x_0\|^2_{\mathbb{R}^m}}{R^2} \right) \mathrm{s}^{\perp}(\xi, \xi) \Bigg\} \\
= -\mathrm{e}^{\sigma t_0} \mathrm{s}^{\perp}(\xi, \xi) \left\{ \frac{\sigma}{R^2} \|x_0\|^2_{\mathbb{R}^m} - 3 \frac{c(T')}{R^2} \|x_0\|_{\mathbb{R}^m} - \frac{c(T')}{R^2} + \sigma \right\} .
\end{aligned}
$$

Choosing $R_0 > 0$ (depending on σ and T') large enough and using the same argument as in the proof of lemma 6.2.4, we obtain the estimate

$$
\mathcal{A} \geq -\mathrm{e}^{\sigma t_0} \mathrm{s}^{\perp}(\xi, \xi) \frac{\sigma}{2} > 0 \qquad \text{for any } x_0 \text{ and for all } R \geq R_0 .
$$

To estimate $\mathcal{B}$, recall that since $\chi_{|(x_0, t_0)}(\xi, \eta) = 0$ for any null-eigenvector ξ and any normal vector η, it is

$$
\varepsilon_2 t_0 \vec{H}_{\xi} \vec{H}_{\eta} = -\mathrm{e}^{\sigma t_0} \phi_R \mathrm{s}^{\perp}(\xi, \eta) .
$$

Let $\{\xi_1, \ldots, \xi_n\}$ denote an orthonormal basis of the normal space $T^{\perp}_{x_0}\mathbb{R}^m$ and choose a g-orthonormal basis $\{e_1, \ldots, e_m\}$, such that it diagonalizes s. Then

$$
\begin{aligned}
\mathcal{B} \geq & -2\mathrm{e}^{\sigma t_0} \phi_R \sum_{i,j=1}^{m} \sum_{k=1}^{n} A_{\xi}(e_i, e_j) A_{\xi_k}(e_i, e_j) \mathrm{s}^{\perp}(\xi_k, \xi) \\
& + 2\mathrm{e}^{\sigma t_0} \phi_R \sum_{i,j,k=1}^{m} A_{\xi}(e_i, e_j) A_{\xi}(e_i, e_k) \mathrm{s}(e_j, e_k) \\
& - 2\varepsilon_2 t_0 \sum_{i,j=1}^{m} \sum_{k=1}^{n} A_{\xi_k}(e_i, e_j) A_{\xi}(e_i, e_j) \vec{H}_{\xi_k} \vec{H}_{\xi} \\
& - \varepsilon_2 \vec{H}^2_{\xi} \\
= & -2\mathrm{e}^{\sigma t_0} \phi_R \sum_{i,j=1}^{m} \sum_{k=1}^{n} A_{\xi}(e_i, e_j) A_{\xi_k}(e_i, e_j) \mathrm{s}^{\perp}(\xi_k, \xi) \\
& + 2\mathrm{e}^{\sigma t_0} \phi_R \sum_{i,j,k=1}^{m} A_{\xi}(e_i, e_j) A_{\xi}(e_i, e_k) \mathrm{s}(e_j, e_k) \\
& + 2\mathrm{e}^{\sigma t_0} \phi_R \sum_{i,j=1}^{m} \sum_{k=1}^{n} A_{\xi_k}(e_i, e_j) A_{\xi}(e_i, e_j) \mathrm{s}^{\perp}(\xi_k, \xi) \\
& - \varepsilon_2 \vec{H}^2_{\xi} \\
= & \, 2\mathrm{e}^{\sigma t_0} \phi_R \sum_{i,j=1}^{m} A^2_{\xi}(e_i, e_j) \mathrm{s}_{jj} - \varepsilon_2 \vec{H}^2_{\xi} .
\end{aligned}
$$

Now we fix ε_2, such that $0 < \varepsilon_2 \leq \frac{2\varepsilon_1}{m}$. Then, using $\phi_R \geq 1$, $\mathrm{e}^{\sigma t_0} \geq 1$, the assumption $\mathrm{s}_{jj} \geq \varepsilon_1 > 0$ and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\mathcal{B} &\geq 2\mathrm{e}^{\sigma t_0}\phi_R\varepsilon_1 \sum_{i,j=1}^{m} A_\xi^2(e_i,e_j) - \varepsilon_2 \left(\sum_{i=1}^{m} A_\xi(e_i,e_i)\right)^2 \\ &\geq 2\varepsilon_1 \left\{\sum_{i=1}^{m} A_\xi^2(e_i,e_i) + \sum_{i\neq j} A_\xi^2(e_i,e_j)\right\} - m\varepsilon_2 \sum_{i=1}^{m} A_\xi^2(e_i,e_i) \\ &\geq 0\,.\end{aligned}$$

Collecting all calculations, at (x_0,t_0) we get

$$(\nabla^\perp_{\partial_t}\chi)(\xi,\xi) - (\Delta^\perp\chi)(\xi,\xi) = \mathcal{A} + \mathcal{B} > 0\,,$$

which contradicts equation (6.3.2). This shows the claim.

Now the lemma follows by first letting $R \to \infty$, then $\sigma \to 0$ and finally $T' \to T$. □

Lemma 6.3.2 implies a decay estimate for the mean curvature vector.

Corollary 6.3.3. *Under the assumptions of lemma 6.3.2, we have* $t\|\vec{H}\|^2 \leq C$ *on* $\mathbb{R}^m \times [0,T)$ *for some constant* C *depending only on* ε_1 *and the dimensions* $\dim \mathbb{R}^m$ *and* $\dim \mathbb{R}^n$.

Proof. Let $\{\xi_1,\ldots,\xi_n\}$ be an orthonormal basis of $T^\perp_{x_0}\mathbb{R}^m$. From lemma 6.3.2, we obtain

$$\varepsilon_2 t\|\vec{H}\|^2 = \varepsilon_2 t \sum_{k=1}^{n} (\vec{H}_{\xi_k})^2 \leq -\sum_{k=1}^{n} \mathrm{s}^\perp(\xi_k,\xi_k) \leq n\,.$$

which establishes the claim. □

Remark 6.3.4. If we allow $\varepsilon_1 \geq 0$ in lemma 6.3.2 (corresponding to a weakly length-decreasing map), we may consider the tensor $\mathrm{s}^\perp + \varepsilon_2\vartheta$. Doing the same calculations, we see that $\mathrm{s}^\perp + \varepsilon_2\vartheta \leq 0$ is preserved under the flow. Therefore, it is $\|\vec{H}\|^2 \leq C$ for some constant $C \geq 0$.

In the following, we will analyze the non-parametric version of the mean curvature flow (see equation (3.1.2)), which was given by

$$\frac{\partial f}{\partial t} = \sum_{i,j=1}^{m} \widetilde{\mathrm{g}}^{ij}\partial^2_{ij} f =: \mathscr{L} f\,, \tag{6.3.3}$$

and where $\widetilde{\mathrm{g}}^{ij}$ denotes the components of the inverse of $\widetilde{\mathrm{g}} := \mathrm{g}_{\mathbb{R}^m} + f^*\mathrm{g}_{\mathbb{R}^n}$. Note that, since the length-decreasing condition is preserved under the mean curvature flow by lemma 6.2.5, the same holds for a solution to the non-parametric equation (6.3.3).

We are now going to derive higher-order estimates for equation (6.1.1), which we accomplish by using a blow-up argument.

Lemma 6.3.5 ([CCH12, Lemma 4.2]). *Let $F(x) = (x, f(x,t))$ be a smooth, graphic solution of equation* (6.1.1) *in $[0,T)$ satisfying the bounded geometry condition. Suppose $\|\mathrm{D}f\| \leq C_1$ and $\|\mathrm{D}^2 f\| \leq C_2$ on $\mathbb{R}^m \times [0,T)$ for some constants $C_1, C_2 \geq 0$. Then for every $l \geq 3$ there exists a constant C_l, such that*

$$\sup_{x\in\mathbb{R}^n} \|\mathrm{D}^l f(x,t)\|^2 \leq C_l$$

for all $t \in [0,T)$.

Proof. Note that (possibly after applying a tangential diffeomorphism) f is a solution of equation (6.3.3). If $\|\mathrm{D}^3 f\| \leq C_3$ in $[0,T)$, then a parabolic bootstrapping argument for the quasilinear equation (6.3.3) gives $\|\mathrm{D}^l f\| \leq C_l$ for $l \geq 4$. It will thus suffice to prove the lemma for $l = 3$, which we do in the following.

Suppose $\|\mathrm{D}^3 f\|^2$ was not bounded on $\mathbb{R}^m \times [0,T)$. By the bounded geometry assumption on F (and therefore on f), there would be a sequence $t_k \to T$, such that

$$2\mu_k := \sup_{x\in\mathbb{R}^m} \|\mathrm{D}^3 f(x,t_k)\|^2 \to \infty \qquad \text{and} \qquad \sup_{\substack{x\in\mathbb{R}^m \\ t\leq t_k}} \|\mathrm{D}^3 f(x,t)\|^2 \leq 2\mu_k < \infty\,.$$

Then there exists a sequence $\{x_k\}$, such that $\|\mathrm{D}^3 f(x_k,t_k)\|^2 \geq \mu_k \to \infty$ for $t_k \to T$. Set $\lambda_k := \mu_k^{1/4}$ and let $(y, f_{\lambda_k,1}(y,s))$ denote the parabolic scaling of $(x, f(x,t))$ by $(\lambda_k, 1)$ at (x_k, t_k) for each k (see section 3.1). Then $f_{\lambda_k,1}(0,0) = 0$, $(\widetilde{\mathrm{D}} f_{\lambda_k,1})(0,0) = 0$, $(y, f_{\lambda_k,1}(y,s))$ is a solution of (6.1.1) for $s \in [-\lambda_k^2 t_k, 0]$ and the functions $f_{\lambda_k,1}$ satisfy the quasilinear parabolic equation (6.3.3),

$$\frac{\partial f_{\lambda_k,1}}{\partial s} = \sum_{i,j=1}^{m} \widetilde{\mathrm{g}}^{ij} (f_{\lambda_k,1})_{ij}\,,$$

where the partial derivatives are understood with respect to the scaled coordinates $\{y^1, \ldots, y^m\}$. Using the assumptions $\|\mathrm{D}f\| \leq C_1$ and $\|\mathrm{D}^2 f\| \leq C_2$, in $\mathbb{R}^m \times [-\lambda_k^2 t_k, 0]$ we get for $f_{\lambda_k,1}$

$$\begin{aligned} \|\widetilde{\mathrm{D}} f_{\lambda_k,1}(y,s)\| &\leq \|\mathrm{D}f(x,t)\| + \|\mathrm{D}f(x_0,t_0)\| \leq 2C_1\,, \\ \|\widetilde{\mathrm{D}}^2 f_{\lambda_k,1}\|^2 &= \frac{\|\mathrm{D}^2 f\|^2}{\lambda_k^2} \leq \frac{C_2^2}{\lambda_k^2} \qquad \xrightarrow{k\to\infty} \qquad 0\,. \end{aligned}$$

Further, using the definition of λ_k and the definition of the sequence (x_k, t_k), we obtain

$$\|\widetilde{\mathrm{D}}^3 f_{\lambda_k,1}\|^2 = \frac{\|\mathrm{D}^3 f\|^2}{\lambda_k^4} = \frac{\|\mathrm{D}^3 f\|^2}{\mu_k} \leq 2$$

and

$$\|(\widetilde{\mathrm{D}}^3 f_{\lambda_k,1})(0,0)\|^2 = \frac{\|\mathrm{D}^3 f(x_k,t_k)\|^2}{\lambda_k^4} = \frac{\|\mathrm{D}^3 f(x_k,t_k)\|^2}{\mu_k} \geq 1\,.$$

Using the Schauder estimate [Lie96, Theorem 4.9] together with the boundedness of $\widetilde{\mathrm{D}} f_{\lambda_k,1}$, we obtain

$$\|\widetilde{\mathrm{D}} f_{\lambda_k,1}\|_{\mathscr{C}^{2+\alpha,1+\frac{\alpha}{2}}(B(x_0,1))} \leq c$$

for any $x_0 \in \mathbb{R}^m$ and some constant $c \geq 0$ which only depends on the first and second derivatives of $f_{\lambda_k,1}$, which are already bounded. By differentiation and iteration, analogous formulae hold for the higher derivatives. Therefore, we see that all $\|\widetilde{\mathrm{D}}^l f_{\lambda_k,1}\|$ are uniformly bounded for $s \in [-\lambda_k^2 t_k, 0]$, $l \geq 4$ and any k. From the evolution equation, we get that the derivatives of $f_{\lambda_k,1}$ with respect to s of any positive order are uniformly bounded as well. Further, by the above estimates, the theorem of Arzelà-Ascoli implies that there is a subsequence of $\{f_{\lambda_k}\}$ converging uniformly on compact subsets in $\mathbb{R}^m \times (-\infty, 0]$ to a smooth solution f_∞ of equation (6.3.3) with

$$\|\widetilde{\mathrm{D}}^2 f_\infty\| = 0 \qquad \text{and} \qquad \|(\widetilde{\mathrm{D}}^3 f_\infty)(0,0)\| \geq 1\,,$$

which is a contradiction. Thus $\|\mathrm{D}^3 f\|$ is bounded in $[0,T)$.

Differentiating equation (6.3.3), we obtain the evolution equation for $\partial_k f$, which is of the form $\partial_t(\partial_k f) = \mathscr{L}(\partial_k f) + L$, where L contains terms of order less than 3 (which are already bounded), so that using the above estimate for the norm we get $\|\mathrm{D}^4 f\| \leq C_4$ for some $C_4 \geq 0$. By further iteration, we conclude that all $\|\mathrm{D}^l f\|$ are uniformly bounded for $t \in [0,T)$ and $l \geq 4$. Again by equation (6.3.3), this implies that the time derivatives of f of any positive order are uniformly bounded as well. □

Let us now show that in the length-decreasing setting we get estimates on all derivatives of the function which defines the graph. The statement is analogous to [CCH12, Lemma 4.3], but does not rely on the Lagrangian geometry.

Lemma 6.3.6. *Let $F(x) = (x, f(x,t))$ be a smooth solution of* (6.1.1) *in* $[0,T)$ *satisfying the bounded geometry condition. Suppose that* $(f^* \mathrm{g}_{\mathbb{R}^n})_{|t=0} \leq (1-\delta)\mathrm{g}_{\mathbb{R}^m}$ *holds for a* *fixed $\delta \in (0,1]$. Suppose further that $\|\vec{H}\| \leq C$ on $\mathbb{R}^m \times [0,T)$ for some constant $C \geq 0$. Then for every $l \geq 1$ there exists a constant C_l, such that*

$$\sup_{x\in\mathbb{R}^n} \|\mathrm{D}^l f(x,t)\|^2 \leq C_l$$

for all $t \in [0,T)$.

Proof. By lemma 6.2.5, the length-decreasing condition is preserved in $[0,T)$, so that the relation $f^*\mathrm{g}_{\mathbb{R}^n} \leq (1-\delta)\mathrm{g}_{\mathbb{R}^m}$ holds in $[0,T)$. This shows the claim for $l = 1$. By lemma 6.3.5, we only need to prove the case $l = 2$. Suppose that the claim was false for $l = 2$. Let

$$\eta(t) := \sup_{\substack{x\in\mathbb{R}^m \\ t'\leq t}} \|\mathrm{D}^2 f(x,t')\| .$$

Then there is a sequence (x_k, t_k) along which we have $\|\mathrm{D}^2 f(x_k,t_k)\| \geq \eta(t_k)/2$ while $\eta(t_k) \to \infty$ as $t_k \to T$. Let $\lambda_k := \eta(t_k)$. For each k let $(y, f_{\lambda_k,0}(y,s))$ be the parabolic scaling of the graph $(x, f(x,t))$ by $(\lambda_k, 0)$ at (x_k,t_k). Then $f_{\lambda_k,0}(y,s)$ is a smooth solution of (6.3.3) on $\mathbb{R}^m \times [-\lambda_k^2 t_k, 0]$. Note that by the definition of $\lambda_k = \eta(t_k)$ it is

$$\begin{aligned} \|\widetilde{\mathrm{D}} f_{\lambda_k,0}\| &= \|\mathrm{D} f\| \leq C_1 , \\ \|\widetilde{\mathrm{D}}^2 f_{\lambda_k,0}\| &= \lambda_k^{-1}\|\mathrm{D}^2 f\| \leq 1 \end{aligned}$$

on $\mathbb{R}^m \times [-\lambda_k^2 t_k, 0]$. Moreover, by the definition of the sequence (x_k,t_k), the estimate

$$\|\widetilde{\mathrm{D}}^2 f_{\lambda_k,0}(0,0)\| = \frac{\|\mathrm{D}^2 f(x_k,t_k)\|}{\lambda_k} = \frac{\|\mathrm{D}^2 f(x_k,t_k)\|}{\eta(t)} \geq \frac{1}{2} \tag{6.3.4}$$

holds. By lemma 6.3.5 we conclude that all the higher derivatives of $f_{\lambda_k,0}$ are uniformly bounded on $\mathbb{R}^m \times [-\lambda_k^2 t_k, 0]$. As in the proof of lemma 6.3.5, there is a subsequence of $f_{\lambda_k,0}$ converging smoothly and uniformly on compact subsets in $\mathbb{R}^m \times (-\infty, 0]$ to a smooth solution $f_{\infty,0}$ of (6.3.3) on $\mathbb{R}^m \times (-\infty, 0]$. Since $\|\vec{H}\|^2 \leq C$ for the graphs $(x, f(x,t))$ by assumption, after rescaling we have

$$\|\vec{H}_{\lambda_k,0}\| \leq \frac{C}{\lambda_k}$$

for the graphs $(y, f_{\lambda_k,0}(y,s))$. It follows that for each s the limiting graph $(y, f_{\infty,0}(y,s))$ must have $\|\vec{H}_{\infty,0}\| = 0$ everywhere, as well as $\widetilde{\lambda}_i^2 := f_{\infty,0}^*\mathrm{g}_{\mathbb{R}^m}(e_i,e_i) \leq 1-\delta$. This in turn implies bounds on the Jacobian of the projection $\pi_{\mathbb{R}^m}$ from the graph $(y, f_{\infty,0}(y,s))$ to $\mathbb{R}^m$,

$$\frac{1}{2^{m/2}} < \star\Omega_{\infty,0} = \frac{1}{\sqrt{\prod_{i=1}^m (1+\widetilde{\lambda}_i^2)}} \leq 1 .$$

Thus, we can apply a Bernstein-type theorem of Wang [Wan03, Theorem 1.1] to conclude that the graph $(y, f_{\infty,0}(y,s))$ is an affine subspace of $\mathbb{R}^m \times \mathbb{R}^n$ (note that for $m \leq 2$, this also follows from [JX99, Theorem 1] and for $m \leq 3$ and $n \geq 2$, this follows from [JXY13, Theorem 1.1]). Therefore, $f_{\infty,0}(y,s)$ has to be a linear map, but this contradicts equation (6.3.4), which (taking the limit $k \to \infty$) implies the estimate $\|\widetilde{\mathrm{D}}^2 f_{\infty,0}(0,0)\| \geq 1/2$. □

We also obtain an estimate for the height of the graphs, which is the analogous result to [CCH12, Lemma 4.4] in the Lagrangian setting (i. e. in the Lagrangian case one may replace f by the derivative $\mathrm{D}u$ of the potential u associated to the Lagrangian graph).

Lemma 6.3.7. *Suppose f is a smooth solution to* (6.3.3) *in $[0,T)$ that satisfies the bounded geometry condition. Then*

$$\sup_{x\in\mathbb{R}^m} \|f(x,t)\|^2 \leq \sup_{x\in\mathbb{R}^m} \|f(x,0)\|^2$$

holds for all $t \in [0,T']$, where $T' \in [0,T)$ is arbitrary.

Proof. Fix any $T' \in [0,T)$. Using (6.3.3), we obtain

$$\frac{\partial}{\partial t}\|f\|^2 = 2\langle f, \partial_t f\rangle = 2\sum_{i,j=1}^m \langle f, \tilde{g}^{ij}\partial^2_{ij} f\rangle = \sum_{i,j=1}^m \tilde{g}^{ij}\partial^2_{ij}\|f\|^2 - 2\sum_{i,j=1}^m \tilde{g}^{ij}\langle \partial_i f, \partial_j f\rangle\,,$$

so that

$$\left(\frac{\partial}{\partial t} - \sum_{i,j=1}^m \tilde{g}^{ij}\partial^2_{ij}\right)\|f\|^2 \leq 0\,.$$

As f satisfies the bounded geometry condition, $\tilde{g}$ is uniformly equivalent to the standard Euclidean metric on $\mathbb{R}^m$ for each $t \in [0,T']$. In local coordinates, the Laplacian associated to $\tilde{g}$ is given by

$$\widetilde{\Delta}\|f\|^2 = \sum_{i,j=1}^m \tilde{g}^{ij}\left(\partial^2_{ij}\|f\|^2 - \sum_{k=1}^m (\partial_k\|f\|^2)\widetilde{\Gamma}^k_{ij}\right),$$

so that it differs only by lower-order terms from the above operator. Also note that $a \in \Gamma(T\mathbb{R}^m)$, given by

$$a := \sum_{i,j=1}^m \tilde{g}^{ij}\left(\widetilde{\nabla}_{\partial_i} - \mathrm{D}_{\partial_i}\right)\partial_j = \sum_{i,j,k=1}^m \tilde{g}^{ij}\widetilde{\Gamma}^k_{ij}\partial_k\,,$$

defines a smooth vector field on $\mathbb{R}^m$, and that the coefficients of a are bounded in $[0,T']$ due to the bounded geometry condition. Thus, we can apply the maximum principle (theorem 2.2.6) to the function $\tilde{h}(x,t) := \|f(x,t)\|^2 - \sup_{x\in\mathbb{R}^m}\|f_0(x)\|^2$ to conclude that $\|f\|^2$ is bounded on $[0,T']$ by $\sup_{x\in\mathbb{R}^m}\|f_0(x)\|^2$. □

6.4 Approximating the Solution

In this section, we construct approximations to the solutions of equation (6.3.3). The construction is essentially the same as in [CCH12, Section 5], but applies to a broader geometrical setting, as we will show.

The aim of the following statements and calculations is to use solutions to the heat equation to approximate the Lipschitz initial data of our original problem. We show that under the assumptions of theorem 6.5.1, there exist solutions with bounded geometry to the problem with the approximated initial data.

Using the *heat kernel* for m-dimensional Euclidean space,

$$K(x,y,t) := \frac{1}{(4\pi t)^{m/2}} \exp\left(-\frac{\|x-y\|^2_{\mathbb{R}^m}}{4t}\right),$$

we define the sequence of functions $f_0^k : \mathbb{R}^m \to \mathbb{R}^n$ by setting

$$f_0^k(x) := \int_{\mathbb{R}^m} f_0(y) K\left(x, y, \frac{1}{k}\right) \mathrm{d}y\,, \tag{6.4.1}$$

where integration is done component-wise.

Lemma 6.4.1 ([CCH12, Lemma 5.1])**.** *Let $f_0 : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz continuous function that satisfies*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^m} f_0^* \mathrm{g}_{\mathbb{R}^n}(x) \leq (1-\delta)\mathrm{g}_{\mathbb{R}^m}$$

for some $\delta \in (0,1]$. Then

(i) $f_0^k \xrightarrow{k\to\infty} f_0$ *in* $\mathscr{C}^{0+\alpha}(B(0,R))$ *for any R and* $0 < \alpha < 1$,

(ii) $(f_0^k)^* \mathrm{g}_{\mathbb{R}^n} \leq \mathrm{g}_{\mathbb{R}^m}$ *for every k,*

(iii) $\sup_{\mathbb{R}^m} \|\mathrm{D}^l f_0^k\| < \infty$ *for every $l \geq 2$ and k.*

Proof. The proof is the same as [CCH12, Proof of Lemma 5.1]. For the first claim, see e. g. [Jos07, Lemma 4.2.2]. Since f_0 is length-decreasing by assumption, it has at most linear growth for $\|x\| \to \infty$. Denoting the derivatives with respect to x (resp. y) with D_x (resp. D_y), we calculate

$$\mathrm{D}_x^l f_0^k(x) = -\int_{\mathbb{R}^m} (\mathrm{D}_y f_0(y)) \mathrm{D}_x^{l-1} K\left(x, y, \frac{1}{k}\right) \mathrm{d}y \qquad \text{for every } l \geq 1\,,$$

where we used that because of the growth at infinity the boundary terms vanish when integrating by parts. Therefore, (ii) and (iii) follow from the normalization of the heat kernel and $f_0^* \mathrm{g}_{\mathbb{R}^n} \leq \mathrm{g}_{\mathbb{R}^m}$. □

The estimates in corollary 6.3.3 and the Bernstein-type theorem of Wang [Wan03, Theorem 1.1] imply that the statement of [CCH12, Lemma 5.2] (after replacing $\mathrm{D}u$ by f) also holds true in our setting. The proof together with some small modifications (which are given below) remains the same.

Lemma 6.4.2 ([CCH12, Lemma 5.2, parts 2. and 3.]). *Consider the sequence $\{f_0^k(x)\}$ as given by equation* (6.4.1), *where the function $f_0 : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous and satisfies*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^m} f_0^* g_{\mathbb{R}^n} \leq (1-\delta) g_{\mathbb{R}^m} \qquad \text{for some} \quad \delta \in (0,1]\,.$$

Then for each k, equation (6.3.3) *has a smooth solution $f^k(x,t)$ on $\mathbb{R}^m \times [0,\infty)$ with initial condition $f_0^k(x)$ such that*

(i) for any compact subset $S \subset \mathbb{R}^m \times [0,\infty)$ and $0 < \alpha < 1$ we have

$$\|f^k\|_{\mathscr{C}^{0+\alpha,\alpha/2}(S)} \leq C_S\,,$$

where C_S is a constant depending only on S,

(ii) for any integers $l, m \geq 0$ and compact subset $G \subset \mathbb{R}^m \times (0,\infty)$ we have

$$\|f^k\|_{\mathscr{C}^{l,m}(G)} \leq C_{l,m,\delta,G}\,,$$

where $C_{l,m,\delta,G}$ is a constant depending only on l, m, δ and G.

We prove the statements corresponding to parts 1. and 4. of [CCH12, Lemma 5.2].

Lemma 6.4.3. *Consider the sequence $\{f_0^k(x)\}$ as given by equation* (6.4.1), *where the function $f_0 : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous and satisfies*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^m} f_0^* g_{\mathbb{R}^n} \leq (1-\delta) g_{\mathbb{R}^m} \qquad \text{for some} \quad \delta \in (0,1]\,.$$

Then for each k, equation (6.3.3) *has a smooth solution $f^k(x,t)$ on $\mathbb{R}^m \times [0,\infty)$ with initial condition $f_0^k(x)$ such that*

(i) $(f^k)^* g_{\mathbb{R}^n} \leq g_{\mathbb{R}^m}$,

(ii) for all $l \geq 2$ we have the estimate

$$t^{l-1} \sup_{x\in\mathbb{R}^m} \|\mathrm{D}^l f^k(x,t)\|^2 < C_{l,\delta}$$

for some constant $C_{l,\delta} \geq 0$ depending only on l and δ.

Proof. Fix some k. By lemma 3.1.2, we see that (6.3.3) has a smooth short-time solution with initial condition f_0^k. Therefore, (6.1.1) has a smooth short-time solution with initial condition $F(x,0) = (x, f_0^k(x))$ on $\mathbb{R}^m \times [0, T_k)$ for some $T_k > 0$. Let us assume that T_k is the largest such time. Then by lemma 6.2.5 and condition (ii) in lemma 6.4.1, we have $(f^k)^* \mathrm{g}_{\mathbb{R}^n} \leq \mathrm{g}_{\mathbb{R}^m}$ in $[0, T_k)$ and thus $\|\mathrm{D} f^k\| \leq m$ in $[0, T_k)$.

Now we want to show $T_k = \infty$. Suppose $T_k < \infty$. By lemma 3.1.2, $f^k(x,t)$ has bounded geometry for each $t \in [0, T_k)$, so that the second fundamental form of $F(x,t)$ and all its covariant derivatives are uniformly bounded for each t. Thus, from from the equation

$$(\nabla_{\partial_t} \mathrm{g})(u,v) = -2\langle \vec{H}, A(u,v)\rangle$$

we obtain $\|\nabla_{\partial_t} \mathrm{g}\| < \infty$ for all $t \in [0, T_k)$ and, applying [Ham82, Lemma 14.2], that g for each $t \in [0, T_k)$ is equivalent to the initial metric $\mathrm{g}_{|t=0}$ and therefore equivalent to the Euclidean metric on $\mathbb{R}^m$ for each $t \in [0, T_k)$. Thus $F(x,t)$ has bounded geometry for each $t \in [0, T_k)$, and by corollary 6.3.3, $t\|\vec{H}\|^2 \leq C_k$ in $[0, T_k)$. In particular, there exists $C_k > 0$, such that $\|\vec{H}\|^2(x,t) \leq C_k$ for $(x,t) \in \mathbb{R}^m \times [T_k/2, T_k)$ (this also follows from remark 6.3.4). Then lemma 6.3.6 implies $\|\mathrm{D}^l f^k\| \leq C_{l,k}$ in $[T_k/2, T_k)$ for all integers $l \geq 1$, which by continuity extends to $[T_k/2, T_k]$. It then follows from lemma 3.1.2 that we can extend the solution beyond T_k. This contradicts the definition of T_k, so that $T_k = \infty$.

To show (ii), first consider $l = 2$, i.e. we show

$$\sup_{x \in \mathbb{R}^m} \|\mathrm{D}^2 f^k(x,t)\|^2 t \leq C_{l,\delta} \tag{6.4.2}$$

for any t and k. Assume this is not the case. Since $f^k(x,t)$ satisfies the bounded geometry condition, it is

$$\sup_{x \in \mathbb{R}^m} \|\mathrm{D}^2 f^k(x,t)\| < \infty$$

for each t. If equation (6.4.2) does not hold, there exists a sequence $t_k > 0$ with

$$\sup_{\substack{x \in \mathbb{R}^m \\ t \leq t_k}} \|\mathrm{D}^2 f^k(x,t)\|^2 t_k = \sup_{x \in \mathbb{R}^m} \|\mathrm{D}^2 f^k(x,t_k)\|^2 t_k =: 2\mu_k \qquad \xrightarrow{k \to \infty} \infty\,. \tag{6.4.3}$$

Further, for each t_k, there exists $x_k \in \mathbb{R}^m$, such that

$$\|\mathrm{D}^2 f^k(x_k, t_k)\|^2 t_k \geq \mu_k\,. \tag{6.4.4}$$

Denote $(y, f_{\lambda_k,0}(y,s))$ be the parabolic scaling of the graphs $(x, f^k(x,t))$ for $t \leq t_k$ by $(\lambda_k, \kappa) := (\sqrt{\mu_k/t_k}, 0)$ at (x_k, t_k) for each k. The rescaled graph also is a solution of equation (6.3.3) for $s \in [-\mu_k, 0]$ and it is

$$\widetilde{\mathrm{D}} f_{\lambda_k,0}(y,s) = \mathrm{D} f^k(x,t)\,, \qquad \|\widetilde{\mathrm{D}}^2 f_{\lambda_k,0}\| = \frac{\|\mathrm{D}^2 f^k\|}{\sqrt{\mu_k/t_k}} \overset{\text{Eq. (6.4.3)}}{\leq} \sqrt{2}$$

on $\mathbb{R}^m \times [-\mu_k, 0]$. Further, by definition of the parabolic scaling, it is

$$f_{\lambda_k,0}(0,0) = 0 \qquad \text{and} \qquad \|\widetilde{\mathrm{D}} f_{\lambda_k,0}\|^2 \leq m\,.$$

It follows from the theorem of Arzelà-Ascoli that there exists a subsequence of $f_{\lambda_k,0}$ which converges on any compact set of $\mathbb{R}^m \times (-\infty, 0]$ to a solution f_∞ of equation (6.3.3) satisfying

$$f_\infty^* g_{\mathbb{R}^n} \leq (1-\delta) g_{\mathbb{R}^m} \quad \text{and} \quad \|\widetilde{\mathrm{D}}^2 f_\infty(0,0)\|^2 = \frac{1}{\mu_k/t_k}\|\mathrm{D}^2 f_\infty(x_k,t_k)\|^2 \overset{\text{Eq. (6.4.4)}}{\geq} 1\,. \tag{6.4.5}$$

Note that for any $f^k(x,t)$ we have $t\|\vec{H}\|^2 \leq C$ for all t and some C independent of k (this follows from lemma 6.2.5 and corollary 6.3.3). For any μ with $0 < \mu < \mu_k$ we therefore have

$$\|\vec{H}_{\lambda_k,0}\|^2 \leq \frac{C}{\mu_k} \qquad \xrightarrow{k\to\infty} 0$$

uniformly on $\mathbb{R}^m \times [-\mu, 0]$. It follows that $\|\vec{H}_\infty\| = 0$, so that $(y, f_\infty(y,s))$ is a minimal graph. Since $\widetilde{\mathrm{D}} f_\infty(y,s) = \mathrm{D} f_\infty(x,t)$, it is

$$f_\infty^* g_{\mathbb{R}^n} \leq (1-\delta) g_{\mathbb{R}^m}$$

for some $\delta \in (0,1]$, and we therefore also know that f_∞ is a strictly length-decreasing map. As in the proof of lemma 6.3.6, we apply the Bernstein-type theorem [Wan03, Theorem 1.1] to obtain that f_∞ has to be a linear map. But this contradicts equation (6.4.5), so that our initial assumption was false and we have proven equation (6.4.2).

Now let $l \geq 3$, and suppose that $\|\mathrm{D}^l f^k\|^2 t^{l-1}$ is not uniformly bounded in k. Then (as above) there exists (x_k, t_k), such that

$$\sup_{\substack{x\in\mathbb{R}^m \\ t\leq t_k}} \|\mathrm{D}^l f^k(x,t)\|^2 t^{l-1} =: 2\sigma_k \qquad \xrightarrow{k\to\infty} \infty \tag{6.4.6}$$

and

$$\|\mathrm{D}^l f^k(x_k,t_k)\|^2 t_k^{l-1} \geq \sigma_k\,. \tag{6.4.7}$$

Let

$$\lambda_k := \sqrt{\frac{\sigma_k^{1/(l-1)}}{t_k}}$$

and denote by $(y, f_{\lambda_k,1}(y,s))$ the parabolic scaling of $(x, f^k(x,t))$ for $t_k/2 \leq t \leq t_k$ by $(\lambda_k, 1)$ at (x_k, t_k) for each k. The rescaled graph $(y, f_{\lambda_k,1}(y,s))$ is a solution to equation (6.3.3) for $s \in [-\sigma_k^{1/(l-1)}, 0]$. We calculate

$$\begin{aligned}\|\widetilde{\mathrm{D}}^2 f_{\lambda_k,1}(y,s)\|^2 &= \lambda_k^{-2}\|\mathrm{D}^2 f^k(x,t)\|^2 = \frac{t_k}{\sigma_k^{1/(l-1)}}\|\mathrm{D}^2 f^k(x,t)\|^2 \\ &\leq \frac{2t}{\sigma_k^{1/(l-1)}}\|\mathrm{D}^2 f^k(x,t)\|^2 \overset{\text{Eq. (6.4.2)}}{\leq} \frac{2C_{l,\delta}}{\sigma_k^{1/(l-1)}}\,.\end{aligned}$$

Using $l \geq 3$ and that $\sigma_k \to \infty$ for $k \to \infty$ by equation (6.4.6), we obtain

$$\|\widetilde{\mathrm{D}}^2 f_{\lambda_k,1}(y,s)\|^2 \xrightarrow{k\to\infty} 0\,.$$

Fix any η with $0 < \eta < \sigma_k^{1/(l-1)}$ for all k. By lemma 6.3.5, all higher derivatives of $f_{\lambda_k,1}$ are uniformly bounded on $\mathbb{R}^m \times [-\eta, 0]$ and by the definition of the parabolic scaling it is $f_{\lambda_k,1}(0,0) = 0$ and $\widetilde{\mathrm{D}} f_{\lambda_k,1}(0,0) = 0$. Using the theorem of Arzelà-Ascoli, we conclude that $f_{\lambda_k,1}$ converges subsequentially on compact sets to a solution f_∞ of equation (6.3.3) on $\mathbb{R}^m \times [-\eta, 0]$. But again, $\|\widetilde{\mathrm{D}}^2 f_\infty\| = 0$ on $\mathbb{R}^m \times [-\eta, 0]$ contradicts equation (6.4.7), which implies $\|\widetilde{\mathrm{D}}^l f_{\lambda_k,1}(0,0)\|^2 \geq 1$ for $l \geq 3$ and any k. □

6.5 Proof of the Theorem

Let us collect the results of the preceding sections to prove the main result of this chapter, as given by the following statement.

Theorem 6.5.1. *Suppose $f_0 : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz continuous function and that there exists a fixed $\delta \in (0,1]$, such that*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^m} f_0^* \mathrm{g}_{\mathbb{R}^n}(x) \leq (1-\delta)\mathrm{g}_{\mathbb{R}^m}(x)\,.$$

Then equation (6.1.1) *with initial condition $F(x,0) := (x, f_0(x))$ has a long-time smooth solution for all $t > 0$, such that the following statements hold.*

(i) Along the flow, the evolving submanifold stays the graph of a strictly length-decreasing map $f_t : \mathbb{R}^m \to \mathbb{R}^n$ for all $t > 0$.

(ii) The mean curvature vector of the graph satisfies the estimate

$$t\|\vec{H}\|^2 \leq C$$

for some constant $C \geq 0$.

(iii) All spatial derivatives of order $k \geq 2$ of f_t satisfy the estimate

$$t^{k-1} \sup_{x \in \mathbb{R}^m} \|\mathrm{D}^k f(x,t)\|^2 \leq C_{k,\delta}$$

for some constant $C_{k,\delta} \geq 0$ depending only on k and δ. In addition,

$$\sup_{x \in \mathbb{R}^m} \|f(x,t)\|^2 \leq \sup_{x \in \mathbb{R}^m} \|f(x,0)\|^2$$

for all $t \in [0,T)$ and the flow has long-time existence.

If f_0 also satisfies $f_0(x) \to 0$ as $\|x\| \to \infty$, then $\|f(x,t)\| \to 0$ smoothly on compact sets of $\mathbb{R}^m$ for $t \to \infty$.

Proof. Let f_0 satisfy the assumptions of the theorem. Further, let $\{f_0^k\}$ be the sequence of approximations of f_0 as given by equation (6.4.1). By lemmas 3.1.2 and 6.4.3, for each k we have a long-time solution $f^k(x,t)$ to (6.3.3).

For any fixed positive R, T, ε, by Arzelà-Ascoli and lemma 6.4.2, there exists a subsequence of $\{f^k\}$, which we still denote by $\{f^k\}$, such that for some function $f_{R,T}$ on $\overline{B(0,R)} \times [0,T]$ we have:

(i) $f^k \to f_{R,T}$ in $\mathscr{C}^{\alpha,\alpha/2}(\overline{B(0,R)} \times [0,T])$ for any $0 < \alpha < 1$ with

$$\|f_{R,T}\|_{\mathscr{C}^{\alpha,\alpha/2}} \leq C_{R,T}\,.$$

(ii) $f^k \to (f_{R,T})_{|[\varepsilon,T]}$ in $\mathscr{C}^{l,m}(\overline{B(0,R)} \times [\varepsilon,T])$ for any l, m with

$$\|f_{R,T,\varepsilon}\|_{\mathscr{C}^{l,m}} \leq C_{l,m,R,T,\varepsilon}\,.$$

Then letting $R \to \infty$, $T \to \infty$, $\varepsilon \to 0$ and using a diagonal subsequence argument, we may conclude that $\{f^k\}$ has a convergent subsequence converging on every compact subset of $\mathbb{R}^m \times [0,\infty)$ to a solution f to (6.3.3) which is smooth on $\mathbb{R}^m \times (0,\infty)$ and $f \in \mathscr{C}^0$ in t at $t = 0$.

If we assume $\|f_0\| \to 0$ for $\|x\| \to \infty$, we know by lemma 6.3.7 that $\sup_{x \in \mathbb{R}^m} \|f(x,t)\|$ stays bounded. As the singular values $\lambda_1^2, \ldots, \lambda_m^2$ are uniformly bounded, so is $\widetilde{\mathrm{g}}$, which means the equation

$$\frac{\partial f}{\partial t} = \sum_{i,j=1}^{m} \widetilde{\mathrm{g}}^{ij} \partial_{ij}^2 f$$

is uniformly parabolic. Then, by the theorem in [Ili61] (see also theorem B.1 in appendix B), $f(x,t) \to 0$ as $t \to \infty$, uniformly with respect to x. This shows the convergence part of theorem 6.5.1. $\square$

Remark 6.5.2. (i) Let us explain the relation between theorem 6.5.1 and [CCH12, Theorem 1.1]. Equip $\mathbb{R}^{2n}$ with Cartesian coordinates $\{x^1, \dots, x^n, y^1, \dots, y^n\}$ and let $\mathrm{J} \in \mathrm{End}(\mathbb{R}^{2n})$ denote the complex structure on $\mathbb{R}^{2n}$ defined via

$$\mathrm{J}\frac{\partial}{\partial x^i} := \frac{\partial}{\partial y^i}\,, \qquad \mathrm{J}\frac{\partial}{\partial y^i} := -\frac{\partial}{\partial x^i}\,, \qquad i = 1, \dots, n\,.$$

Then $\nabla^{\langle\cdot,\cdot\rangle}\mathrm{J} = 0$ and

$$\mathrm{s}_{\mathbb{R}^n \times \mathbb{R}^n}(\xi, \eta) = -\mathrm{s}_{\mathbb{R}^n \times \mathbb{R}^n}(\mathrm{J}\xi, \mathrm{J}\eta) \qquad \text{for all} \quad \xi, \eta \in \Gamma(T\mathbb{R}^{2n})\,.$$

Now assume that $F := \mathrm{id}_{\mathbb{R}^n} \times f : \mathbb{R}^n \to \mathbb{R}^{2n}$ is a Lagrangian immersion, i. e. with respect to the product metric $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{2n}$ it is $\langle \mathrm{J}\mathrm{d}F(u), \mathrm{d}F(v)\rangle = 0$ for all $u, v \in \Gamma(T\mathbb{R}^n)$. Then the complex structure J provides an isomorphism between the normal bundle and the tangent bundle, and we can use it to map normal vectors to tangent vectors. For normal vectors, we also have the equality

$$\mathrm{s}^{\perp}(\xi, \eta) = \mathrm{s}_{\mathbb{R}^n \times \mathbb{R}^n}(\xi, \eta) = -\mathrm{s}_{\mathbb{R}^n \times \mathbb{R}^n}(\mathrm{J}\xi, \mathrm{J}\eta) \qquad \text{for all} \quad \xi, \eta \in \Gamma(T^{\perp}\mathbb{R}^{2n})\,.$$

Thus, in the Lagrangian case one can consider the tensor $-\mathrm{s}_{\mathbb{R}^n \times \mathbb{R}^n}(\mathrm{J}\cdot, \mathrm{J}\cdot)$ instead of $\mathrm{s}^{\perp}$ as done by Chau, Chen and He.

(ii) Let M be a $2n$-dimensional Kähler-Einstein manifold that is either compact or complete with bounded curvature quantities. Further, let $F : L \to M$ be a compact immersion of L as a Lagrangian submanifold. In this case, the quantity $\mathrm{s} - \varepsilon\widetilde{\theta}$ with $\widetilde{\theta}$ given by

$$\widetilde{\theta} \in \mathrm{Sym}(T^*M \otimes T^*M)\,, \qquad \widetilde{\theta}(v, w) := \langle \vec{H}, \mathrm{J}\mathrm{d}F(v)\rangle \langle \vec{H}, \mathrm{J}\mathrm{d}F(w)\rangle$$

was previously considered in [Smo04] to obtain bounds on the mean curvature vector.

Chapter 7
Conclusion and Directions

In the previous chapters, we studied the behavior of maps f between manifolds that evolve under the mean curvature flow. We analyzed tensors and functions composed of the semi-Riemannian metric $\mathrm{s}_{M\times N}$, its restriction to the tangent bundle and to the normal bundle of the graph $\Gamma(f)$, the induced metric g, the mean curvature vector $\vec{H}$ and the second fundamental form A of the graph.

In the compact, two-dimensional setting we obtained estimates for polynomials symmetric in the eigenvalues of pulled-back tensor s. These estimates were used to control the mean curvature vector and the second fundamental form. We showed that the mean curvature vector of a strictly area-decreasing map decays at least as t^{-1} if the curvatures of the surfaces could be separated by a positive constant $\sigma > 0$. Further, we showed that if the curvature of the target and its derivatives are bounded and the differential of the initial map satisfies a stronger condition, the second fundamental form also falls off at least as t^{-1}.

In the non-compact setting, we considered length-decreasing maps between Euclidean spaces. In this setting, we analyzed the behavior of the restriction of $\mathrm{s}_{M\times N}$ to the normal bundle to control the mean curvature vector. Using a blow-up argument, we showed that all higher derivatives of the evolving map decay at a certain rate in time. Further, if the initial map approaches zero at infinity, it converges to the zero map.

The Compact Case. Let us comment on the assumptions on the maps and the curvatures made in the theorems 5.2.4 and 5.3.12.

(i) We state an observation made in [SS14a], which also applies to our case.

Consider a smooth map f between two Riemann manifolds (M, g_M) and (N, g_N), subject to the condition $f^*g_N < cg_M$ for some $c > 0$. If M is compact, such a constant always exists. Define the rescaled metrics

$$\widetilde{g}_M := cg_M\,, \qquad \widetilde{g}_N := c^{-1}g_N\,.$$

Then f is length-decreasing with respect to the metrics $\widetilde{g}_M$ and g_N as well as with respect to the metrics g_M and $\widetilde{g}_N$. This means that by appropriately rescaling the target or the domain, any smooth map can be made length-decreasing.

(ii) An interesting case is when the map is weakly area-decreasing but not area-preserving. In this case and under certain curvature assumptions, one may expect that for times $t > 0$, the map immediately becomes strictly area-decreasing and the theorems may be applicable.

(iii) The decay estimate for the second fundamental form relies on a smallness assumption on the singular values of the differential of the defining map. It would be interesting to see if these constraints can be removed, e. g. by considering the evolution of other geometrical quantities.

(iv) The theorems only hold if $\dim M = \dim N = 2$, so one may try to obtain similar results in the higher-dimensional as well as in the higher-codimensional case. Here, one may have to consider other functions or tensors, since our estimates (for example when considering the evolution equation for the trace of s) do not hold anymore.

(v) Recall that the curvature assumptions included the condition

$$\sec_N \leq \sigma \leq \sec_M \qquad \text{with} \qquad \sigma > 0$$

or (see remark 5.2.5)

$$\sec_N \leq 0 = \sec_M\,.$$

It is not clear if the positivity of σ is essential or if similar versions of the theorems also hold in the case $\sigma \leq 0$.

The Non-Compact Case. We make some observations with respect to the assumptions in theorem 6.5.1.

(i) The rescaling procedure stated for the compact case also applies to the case where the manifolds are Euclidean spaces. Also, we note that the theorem does not apply for weakly length-decreasing maps which are contain points where they are length-preserving.

(ii) When extending this result to the case of non-flat manifolds, one needs to construct the appropriate cut-off functions to adapt the strategy of the given proof. Also, to show convergence, one may need to modify theorem B.1 to be applicable to non-flat manifolds.

Appendix A
Solutions to the Differential Equations

We derive the solutions to the differential equations which were used to control $\mathrm{Tr}(s)$, $\|\vec{H}\|^2$ and $\|A\|^2$ in chapter 5.

A.1 The Equation $\partial_t \ln u = \frac{\sigma}{4}(4 - u^2)$

We separate the variables to obtain

$$\int_{u_0}^{u} \frac{\mathrm{d}\widetilde{u}}{\widetilde{u}(2-\widetilde{u})(2+\widetilde{u})} = \frac{\sigma}{4}\int_{t_0}^{t} \mathrm{d}\widetilde{t} = t - t_0 \,.$$

The partial fraction decomposition of the left-hand side is given by

$$\begin{aligned}
&\int_{u_0}^{u} \left[\frac{1}{4\widetilde{u}} + \frac{1}{8(2-\widetilde{u})} - \frac{1}{8(2+\widetilde{u})}\right] \mathrm{d}\widetilde{u} \\
&\qquad = \frac{1}{4}\ln u - \frac{1}{8}\ln(2-u) - \frac{1}{8}\ln(2+u) + \text{const} \\
&\qquad = \frac{1}{4}\ln u - \frac{1}{4}\ln\sqrt{4-u^2} + \text{const} = \frac{1}{4}\ln\frac{u}{\sqrt{4-u^2}} + \text{const}\,,
\end{aligned}$$

where all constants appearing in the calculation are collected in the respective last term. Solving the resulting equation subject to the constraint $u > 0$, we get

$$u(t) = \frac{2\exp(\sigma t)}{\sqrt{c_1 + \exp(2\sigma t)}}$$

for some constant $c_1 > 0$.

A.2 The Equation $\partial_t \ln h = \kappa_M\left(1 - \frac{\exp(2\sigma t)-c_1}{\exp(2\sigma t)+c_1}\right)$

We integrate the equation with respect to t. Using the substitution $x := \exp(2\sigma t)$, we obtain

$$\int \frac{\exp(2\sigma t) - c_1}{\exp(2\sigma t) + c_1}\mathrm{d}t = \frac{1}{2\sigma}\int \frac{x - c_1}{x + c_1}\frac{\mathrm{d}x}{x}\,.$$

By partial fraction decomposition, we further get

$$\frac{1}{2\sigma}\int \frac{x - c_1}{x + c_1}\frac{\mathrm{d}x}{x} = \frac{1}{2\sigma}\int\left[-\frac{1}{x} + \frac{2}{x + c_1}\right]\mathrm{d}x = \frac{1}{2\sigma}\Big[-\ln x + 2\ln(x + c_1)\Big]\,.$$

Substituting back, we have

$$\begin{aligned}\frac{1}{2\sigma}\Big[-\ln x + 2\ln(x + c_1)\Big] &= -t + 2\ln\Big(\exp(2\sigma t) + c_1\Big)\\ &= \ln\frac{(\exp(2\sigma t) + c_1)^{1/\sigma}}{\exp(t)}\,.\end{aligned}$$

Therefore, the integrated equation reads

$$\begin{aligned}\ln h &= \kappa_M\left(t - \ln\frac{(\exp(2\sigma t) + c_1)^{1/\sigma}}{\exp(t)}\right) + \widetilde{c}_2\\ &= \ln\frac{\exp(2\kappa_M t)}{(\exp(2\sigma t) + c_1)^{\kappa_M/\sigma}} + \widetilde{c}_2\,.\end{aligned}$$

Solving for h, we obtain the solution of the original equation.

A.3 The Inequality $\frac{1}{x}pp' + 2pp'' - (p')^2 \leq 0$

We try to find a positive function p, defined on the interval $(\varepsilon, 2]$ for some $\varepsilon \in [0, 2)$, that satisfies this inequality. As a first step, let us consider the corresponding differential equation

$$\frac{1}{x}pp' + 2pp'' - (p')^2 = 0\,.$$

Note that $p \mapsto \kappa p$ for $\kappa \in \mathbb{R}$ maps solutions onto other solutions. Let us set $p(x) := [g(x)]^\alpha$ for some $\alpha \in \mathbb{R}$ to be chosen later. Then

$$p'(x) = \alpha[g(x)]^{\alpha-1}g'(x) \qquad \text{and} \qquad p''(x) = \alpha[g(x)]^{\alpha-2}\Big((g'(x))^2 + g(x)g''(x)\Big)\,.$$

Inserting these into the equation, we obtain

$$\begin{aligned} 0 &= \frac{\alpha}{x}[g(x)]^{2\alpha-1}g'(x) + 2\alpha[g(x)]^{2\alpha-2}\Big((g'(x))^2 + g(x)g''(x)\Big) \\ &\quad - \alpha^2[g(x)]^{2\alpha-2}(g'(x))^2 \\ &= \alpha[g(x)]^{2\alpha-1}\left(\frac{g'(x)}{x} + 2g''(x)\right) + \alpha[g(x)|^{2\alpha-2}(g'(x))^2(2-\alpha)\,. \end{aligned}$$

Choosing $\alpha := 2$ and assuming $g'(x) \neq 0$, we therefore need to solve the equation

$$\frac{g'(x)}{x} + 2g''(x) = 0 \qquad \Leftrightarrow \qquad \frac{g''(x)}{g'(x)} = -\frac{1}{2x}\,.$$

By integration, we obtain the solution for $x > 0$ as

$$g'(x) = \frac{c}{\sqrt{x}} \qquad \Leftrightarrow \qquad g(x) = 2c\sqrt{x} + d\,,$$

where $c, d \in \mathbb{R}$ are constants and, to satisfy the assumption, $c \neq 0$. In conclusion, a solution to the initial differential equation is given by

$$p(x) = \left(2c\sqrt{x} + d\right)^2.$$

Using $c \neq 0$ and the scaling property mentioned above, the (positive) solution to the equation may be written as

$$p(x) = c_2\left(c_1 + \sqrt{x}\right)^2, \qquad c_1 \in \mathbb{R}\,, \quad c_2 > 0\,.$$

Note that the exponents in this function, $\alpha := 1/2$ and $\beta := 2$, satisfy the relation $\alpha\beta = 1$. Let us define the modified function

$$p_k(x) := c_2\left(c_1 + x^{1/k}\right)^k, \qquad c_1 \in \mathbb{R}\,, \quad c_2 > 0\,, \quad k > 0\,.$$

Then

$$p_k'(x) = c_2\left(c_1 + x^{1/k}\right)^{k-1} x^{(1-k)/k} = p_k(x)\frac{x^{(1-k)/k}}{c_1 + x^{1/k}}$$

and

$$
\begin{aligned}
p_k''(x) &= p_k(x)\left(\frac{x^{(1-k)/k}}{c_1 + x^{1/k}}\right)^2 + p_k(x)\frac{\frac{1-k}{k}x^{(1-2k)/k}(c_1 + x^{1/k}) - \frac{1}{k}x^{2(1-k)/k}}{(c_1 + x^{1/k})^2} \\
&= \frac{k-1}{k}\frac{p_k(x)}{(c_1 + x^{1/k})^2}\left(x^{2(1-k)/k} - x^{(1-2k)/k}c_1 - x^{2(1-k)/k}\right) \\
&= p_k(x)\frac{1-k}{k}\frac{c_1 x^{(1-2k)/k}}{(c_1 + x^{1/k})^2}\,.
\end{aligned}
$$

Inserting this into the differential inequality, we obtain the condition

$$
\begin{aligned}
0 &\geq p_k^2(x)\left(\frac{x^{(1-2k)/k}}{c_1 + x^{1/k}} + 2\frac{1-k}{k}\frac{c_1 x^{(1-2k)/k}}{(c_1 + x^{1/k})^2} - \frac{x^{2(1-k)/k}}{(c_1 + x^{1/k})^2}\right) \\
&= p_k^2(x)c_1\frac{2-k}{k}\frac{x^{(1-2k)/k}}{(c_1 + x^{1/k})^2}\,.
\end{aligned}
$$

Since $p_k^2(x) \geq 0$ and $x > 0$ by assumption, it follows that either $c_1 > 0$ and $k \geq 2$ or $c_1 < 0$ and $k \in (0, 2]$.

Appendix B
The Convergence Theorem of Il'in

We give a translation of the proof of the theorem found in [Ili61], which was used in the proof of theorem 6.5.1.

Let $a(x,t) \in \mathrm{Mat}(\mathbb{R}, m)$ be positive definite for any (x,t) and consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{m} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u\,, \qquad u(x,1) = u_1(x)\,. \tag{B.1}$$

Note that the entries of $a(x,t)$ may also depend on the function u and its derivatives $\mathrm{D}u$. We will give a proof of the following statement.

Theorem B.1 ([Ili61]). *Let u be a solution to the Cauchy problem* (B.1) *and assume that $u(x,1) \to 0$ for $\|x\| \to \infty$. Further, assume that equation* (B.1) *is uniformly parabolic for all $x \in \mathbb{R}^m$ and $t \geq 1$, i. e. there exist $C_1, C_2 \in \mathbb{R}^{>0}$ with*

$$C_1 \|\xi\|^2 \leq \langle a\xi, \xi \rangle \leq C_2 \|\xi\|^2\,. \tag{B.2}$$

Also suppose that the coefficients $b_i(x,t)$ are bounded for $1 \leq t \leq T$ for all $T > 1$, $c(x,t) \leq 0$ and that there exists $\delta \geq 0$ and $r_0 > 0$ with

$$\langle b(x,t), x \rangle > \delta - \mathrm{Tr}(a(x,t)) \qquad \text{for} \quad t \geq 1 \text{ and } \|x\| \geq r_0\,. \tag{B.3}$$

Moreover, assume that there exists $N \geq 0$ with

$$\|b(x,t)\| \leq N \qquad \text{for} \quad \|x\| \leq r_0\,. \tag{B.4}$$

Then $u(x,t) \xrightarrow{t \to \infty} 0$ uniformly with respect to x.

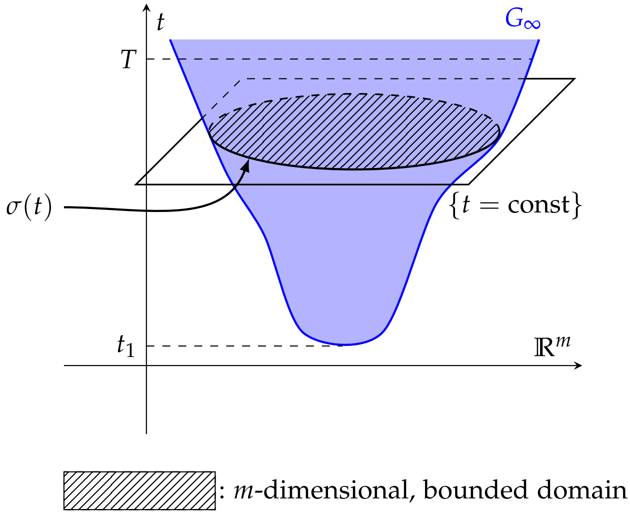

Figure B.1: The definitions of G_∞ and $\sigma(t)$. The part of G_∞ between t_1 (included) and T (excluded) is denoted by G_T.

Let us introduce some notation. Fix some $t_1 \geq 0$. We define the half-space

$$\mathcal{H}_{t_1} := \mathbb{R}^m \times \{t \geq t_1\} \qquad \subset \mathbb{R}^m \times \mathbb{R}\,.$$

Let $G_\infty \subset \mathbb{R}^m \times \{t \in \mathbb{R} : t \geq t_1 \geq 0\}$ be a $(m+1)$-dimensional region in the half-space $\mathcal{H}_{t_1}$, such that its section with the planes $\{t = \text{const}\}$ for $t \geq t_1$ are m-dimensional bounded domains with boundary $\sigma(t)$, and assume that $\sigma(t)$ depends continuously on t. The set of all $\sigma(t)$ for $t \geq t_1$ will be denoted by S and is called the lateral surface area of G_∞. Further, let

$$G_T := G_\infty \cap (\mathbb{R}^m \times \{t \geq t_1 : t < T\})$$

and denote the lateral surface area of G_T together with the lower base by

$$\Gamma_T := \{\sigma(t) : t_1 \leq t < T\} \cup (G_\infty \cap (\mathbb{R}^m \times \{t = t_1\}))\,.$$

The proof will be based on the following three lemmas.

Lemma B.2 ([Ili61, Lemma 1]). *Assume that $u(x,t)$ is continuous in G_T and differentiable everywhere but in a finite number of continuously differentiable surfaces R_k in G_T. Assume that outside the surfaces R_k, equation* (B.1) *is satisfied, i. e.*

$$Lu := \sum_{i,j=1}^{m} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t} \leq 0\,, \tag{B.5}$$

where $a_{ij}(x,t)$ are the entries of a positive definite matrix and $c(x,t) \leq 0$. On the surfaces R_k, assume that the one-sided normal derivatives of $u(x,t)$ exist, and that they *satisfy relation*

$$\frac{\partial u}{\partial n_+} < \frac{\partial u}{\partial n_-}, \tag{B.6}$$

where $\frac{\partial u}{\partial n_+}$ *denotes the one-sided derivative with respect to the normal* n *of the surface* R_k *and* $\frac{\partial u}{\partial n_-}$ *denotes the one-sided derivative with respect to the normal* $-n$.

Then, if $u(x,t) \geq 0$ *in* Γ_T*, it is* $u(x,t) \geq 0$ *in* $\overline{G}_T$.

Proof. Suppose, on the contrary, that $u(x,t)$ attains a negative minimum in G_T. Then, for sufficiently small positive ε, the function $u(x,t) + \varepsilon t$ also achieves its minimum in G_T. By assumption (B.6), the minimum cannot be attained at least on the surfaces R_k. But in the rest of G_T we have $L(u + \varepsilon t) \leq -\varepsilon < 0$ by (B.5) and at the minimum of $u + \varepsilon t$ also $L(u + \varepsilon t) \geq 0$, which is a contradiction. This proves the lemma. □

Lemma B.3 ([Ili61, Lemma 2]). *Let* $u(x,t)$ *be bounded and continuous in* $t \geq 1$*, such that* $\|u(x,t)\| \leq M_1$ *for all* $x \in \mathbb{R}^m$ *and* $t \geq 1$*. Further, assume that* u *is twice continuously differentiable and satisfies* (B.5) *everywhere except at a finite number of continuously differentiable surfaces* R_k*, where* $a(x,t)$ *is a positive definite matrix,* $c(x,t) \leq 0$*, and the coefficients* $a_{ij}(x,t)$ *and* $b_i(x,t)$ *are bounded for any* $1 \leq t \leq T$ *with* $T > 1$ *arbitrary. On the surfaces* R_k*, assume that inequality* (B.6) *holds and also that* $u(x,1) > 0$*. Then* $u(x,t) > 0$ *for any* $x \in \mathbb{R}^m$ *and* $t \geq 1$.

Proof. Fix an arbitrary point (x_0, t_0) and set

$$w(x,t) := u(x,t) + M_1 \frac{\cosh r}{\cosh l} e^{\beta t}, \qquad r := \|x\|,$$

where $l > \|x_0\|$ and $\beta > 0$ will be chosen such that $Lw < 0$ for $t > 1$. This is possible, since

$$\begin{aligned} Lw = Lu + \frac{M_1}{\cosh l} \cosh(r) e^{\beta t} \Bigg\{ & \frac{1}{r} \sum_{i=1}^m a_{ii} \tanh(r) \\ & + \frac{1}{r^2} \sum_{i,j=1}^m a_{ij} x_i x_j \left(1 - \frac{\tanh(r)}{r}\right) + \tanh(r) \sum_{i=1}^m b_i \frac{x_i}{r} + c - \beta \Bigg\} < 0 \end{aligned}$$

for sufficiently large β.

Note that $w(x,t) > 0$ for $t = 1$. On the boundary of $G_{t_0} = \{1 < t \leq t_0, r < l\}$ it is $w(x,t) > 0$. Hence, by lemma B.2, it is $w(x,t) \geq 0$ in G_{t_0}, which means

$$u(x_0, t_0) \geq -\frac{M_1 \cosh \|x_0\|}{\cosh l} e^{\beta t_0}.$$

As $l > \|x_0\|$ is arbitrary, by taking the limit $l \to \infty$ it follows that $u(x_0, t_0) \geq 0$. Since the point (x_0, t_0) was arbitrary, the lemma is proven. □

Lemma B.4 ([Ili61, Lemma 3]). *Suppose the function $u_1(x,t)$ satisfies the conditions* (B.5) *and* (B.6) *of lemma B.2 in G_∞ and additionally $u_1(x,t)_{|S} \geq 0$. Also assume that there is a continuous, positive, bounded function $v(x,t)$ defined on G_∞, such that the relation*

$$Lv < -\frac{\gamma v}{t}$$

holds in G_∞ for some constant $\gamma > 0$ everywhere except for finitely many continuously differentiable surfaces on which the condition (B.6) *holds. Then it is*[1] $\liminf_{t\to\infty} \min_x u(x,t) \geq 0$.

Proof. Consider the function

$$w(x,t) := M_1 \frac{v(x,t)}{t^\gamma} + u_1(x,t)\,,$$

where M_1 is the upper limit of the function

$$\frac{u_1(x,t_1)}{v(x,t_1)} t_1^\gamma\,.$$

Then we have $w(x,t_1) \geq 0$ and

$$Lw = \frac{M_1 Lv}{t^\gamma} + \frac{\gamma M_1 v}{t^{\gamma+1}} + Lu_1 < 0$$

in G_∞. From lemma B.2 we conclude $w(x,t) \geq 0$ in G_∞, so that $u_1(x,t) \geq -M_1 \frac{v}{t^\gamma}$. This shows the claim. □

Proof of Theorem B.1. We first show that for any $\varepsilon > 0$, there exists a constant $M_1 > 0$, such that

$$\|u(x,t)\| < \varepsilon \qquad \text{for} \quad r \geq M_1\sqrt{t}\,. \tag{B.7}$$

For this we apply the operator

$$L := \sum_{i,j=1}^m a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^m b_i \frac{\partial}{\partial x_i} + c - \frac{\partial}{\partial t}$$

to the auxiliary function

$$v_1(x,t) := \begin{cases} \exp\left(-\alpha \frac{r^2}{t}\right), & r \geq r_0\,, \\ \exp\left(-\alpha \frac{r_0^2}{t}\right), & r \leq r_0\,, \end{cases}$$

[1] The minimizing is done over $G_\infty \cap (\mathbb{R}^m \times \{t\})$.

with a constant $\alpha > 0$. Using equations (B.3) and (B.4) and setting $\alpha := 1/(4C_2)$, we calculate for $r > r_0$

$$
\begin{aligned}
Lv_1 &= \left\{ 4\alpha^2 \sum_{i,j=1}^{m} a_{ij} \frac{x_i x_j}{t^2} - \frac{2\alpha}{t} \sum_{i=1}^{m} a_{ii} - \frac{2\alpha}{t} \sum_{i=1}^{m} b_i x_i + c - \alpha \frac{r^2}{t^2} \right\} \exp\left(-\alpha \frac{r^2}{t} \right) \\
&< \frac{\alpha}{t^2} \mathrm{e}^{-\alpha r^2/t} \left\{ 4\alpha \sum_{i,j=1}^{m} a_{ij} x_i x_j - r^2 \right\} \\
&\leq 0 \,.
\end{aligned}
$$

On the other hand, for $r < r_0$ it is

$$
Lv_1 = -\alpha \frac{r_0^2}{t^2} \exp\left(-\alpha \frac{r_0^2}{t} \right) < 0 \,.
$$

Let us set

$$
w(x,t) := M_2 v_1(x,t) + \frac{\varepsilon}{2} \pm u(x,t) \,,
$$

where the constant $M_2 > 0$ is chosen so that $w(x,1) \geq 0$. Since $u(x,1) \xrightarrow{r\to\infty} 0$, such a constant exists. From the above calculations, for $r \neq r_0$ we have

$$
Lw = M_2 L v_1 + c\frac{\varepsilon}{2} < 0 \,.
$$

Furthermore, on the surface $r = r_0$, the one-sided derivatives of the function $v_1(x,t)$ satisfy condition (B.6), so that by lemma B.3 we have $w(x,t) \geq 0$ for $t \geq 1$. Therefore, choosing $M_1 > 0$ such that $M_2 \mathrm{e}^{-M_1^2} < \frac{\varepsilon}{2}$ shows equation (B.7).

We proceed with the proof of the theorem. Let us define

$$
G_\infty := \{ x : r < M_1 \sqrt{t} \} \times \{ t : t > t_1 \} \,.
$$

On G_∞ we construct a continuous bounded function $v(x,t)$ satisfying the conditions (B.5) and (B.6) of lemma B.2 and the inequality

$$
Lv < -\frac{\gamma_1}{t} \tag{B.8}
$$

for some constant $\gamma_1 > 0$ to be chosen later. We will also fix $t_1 > 0$ in the above definitions later. Let $A := [\cosh(1)]^{-1}$ and

$$
v(x,t) := \begin{cases} 1 - A\left(\cosh \frac{r}{M_1\sqrt{t}} - \cosh \frac{r_0}{M_1\sqrt{t}} \right) \,, & r \geq r_0 \,, \\ 1 + \alpha \frac{\cosh(\beta r_0) - \cosh(\beta r)}{t} \,, & r \leq r_0 \,, \end{cases}
$$

where $\alpha > 0$ and $\beta > 0$ are constants to be chosen later. For $r < r_0$ we calculate

$$\begin{aligned}
Lv &= -\frac{\alpha}{t}\left\{\frac{\beta}{r}\sinh(\beta r)\left(\sum_{i=1}^{m} a_{ii} - \sum_{i,j=1}^{m} a_{ij}\frac{x_i x_j}{r^2}\right) + \frac{\beta^2}{r^2}\sum_{i,j=1}^{m} a_{ij}x_i x_j \cosh(\beta r)\right.\\
&\qquad\left. + \frac{\beta}{r}\sinh(\beta r)\sum_{i=1}^{m} b_i x_i - \frac{\cosh(\beta r_0) - \cosh(\beta r)}{t}\right\} + cv\\
&\le -\frac{\alpha}{t}\left\{\beta^2\cosh(\beta r)\sum_{i,j=1}^{m} a_{ij}\frac{x_i x_j}{r^2} + \frac{\beta}{r}\sinh(\beta r)\sum_{i=1}^{m} b_i x_i - \frac{\cosh(\beta r_0)}{t_1}\right\}\\
&\le -\frac{\alpha}{t}\left\{\beta\cosh(\beta r)\Big(C_1\beta - Nm\tanh(\beta r)\Big) - \frac{\cosh(\beta r_0)}{t_1}\right\}.
\end{aligned}$$

We choose $\beta > \frac{2Nm}{C_1}$ and $\frac{1}{t_1} < \frac{\beta^2 C_1}{4\cosh(\beta r_0)}$. Then it is[2]

$$Lv < -\frac{\alpha\beta^2 c}{4}\frac{1}{t} \qquad \text{for} \quad r < r_0 \quad \text{and} \quad t \ge t_1 .$$

Consequently, inequality (B.8) holds for $t \ge t_1$ and $r < r_0$. The constant $\alpha > 0$ can be chosen sufficiently small, such that condition (B.6) holds, i. e.

$$\lim_{\substack{r\to r_0\\ r<r_0}} \frac{\partial v}{\partial r} = -\alpha\frac{\beta\sinh(\beta r_0)}{t} > \lim_{\substack{r\to r_0\\ r>r_0}} \frac{\partial v}{\partial r} = -\frac{A}{M_1\sqrt{t}}\sinh\frac{r_0}{M_1\sqrt{t}} .$$

In fact, it suffices to set

$$\alpha := \frac{A r_0}{M_1^2\beta\sinh(\beta r_0)} .$$

We verify that inequality (B.8) holds for $r > r_0$,

$$\begin{aligned}
Lv &= -\frac{A}{M_1 r\sqrt{t}}\sinh\frac{r}{M_1\sqrt{t}}\left(\sum_{i=1}^{m} b_i x_i + \sum_{i=1}^{m} a_{ii}\right)\\
&\quad - \frac{A}{M_1 r\sqrt{t}}\frac{\partial}{\partial r}\left(\frac{\sinh\frac{r}{M_1\sqrt{t}}}{r}\right)\sum_{i,j=1}^{m} a_{ij}x_i x_j\\
&\quad - \frac{A}{2M_1 t\sqrt{t}}\left(r\sinh\frac{r}{M_1\sqrt{t}} - r_0\sinh\frac{r_0}{M_1\sqrt{t}}\right)\\
&\le -\frac{A}{M_1 r\sqrt{t}}\sinh\frac{r}{M_1\sqrt{t}}\left(\sum_{i=1}^{m} a_{ii} + \sum_{i=1}^{m} b_i x_i\right)
\end{aligned}$$

[2] Using $\cosh(x) \ge 1$, $\tanh(x) \le 1$ and the estimate for β, the first sum is bounded from above by $-\frac{\alpha}{2t}C_1\beta^2$. Applying the estimate for t_1 shows the claim..

$$< -\frac{A\delta}{M_1 r\sqrt{t}} \sinh \frac{r}{M_1\sqrt{t}}$$
$$< -\frac{A\delta}{M_1^2}\frac{1}{t}\,.$$

Thus, the inequality (B.8) holds everywhere in G_∞ except for the surface $r = r_0$, where the condition (B.6) is satisfied. Since the function $v(x,t)$ is bounded in G_∞, there exists a positive constant $\gamma > 0$, such that $Lv < -\frac{\gamma v}{t}$ is satisfied everywhere in G_∞ except for the surface $r = r_0$. We now apply lemma B.4 to the functions $v(x,t)$ and $u_1(x,t) := \varepsilon \pm u(x,t)$. As shown above, it is $u_1(x,t) \geq 0$ on the boundary $S = \{r = M_1\sqrt{t}\}$ and $Lu_1 = c\varepsilon \leq 0$. Therefore, the lemma implies

$$\liminf_{t\to\infty} \min_{r<M_1\sqrt{t}} \Big(\varepsilon \pm u(x,t)\Big) \geq 0\,,$$

which means that there exists t_2, such that for $t > t_2$ and $r \leq M_1\sqrt{t}$ the inequality

$$\varepsilon \pm u(x,t) \geq -\varepsilon \qquad \Leftrightarrow \qquad \|u(x,t)\| < 2\varepsilon$$

holds. This, together with equation (B.7), proves the theorem. $\square$

Appendix C
Parabolic Hölder Spaces

In this short appendix, for a reference we recall the notion of (parabolic) Hölder spaces. We follow [Lie96, Chapter IV.1].

We use the notation $X = (x, t) \in \mathbb{R}^{n+1}$ and $X_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$. The norm on $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ are given by

$$\|x\|^2 := \left(\sum_{k=1}^{m} (x^i)^2\right) \qquad \text{and} \qquad \|X\| := \max\{\|x\|, |t|^{1/2}\},$$

respectively.

For $\alpha \in (0, 1]$, we say that a function f defined on $\Omega \subset \mathbb{R}^{n+1}$ is *Hölder continuous with exponent* α if the quantity

$$[f]_{\alpha,X_0} := \sup_{X \in \Omega \setminus \{X_0\}} \frac{\|f(X) - f(X_0)\|}{\|X - X_0\|^{\alpha}}$$

is finite. If the semi-norm

$$[f]_{\alpha,\Omega} := \sup_{X_0 \in \Omega} [f]_{\alpha,X_0}$$

is finite, we say that f is *uniformly Hölder continuous in* Ω, and if f is uniformly Hölder continuous on any Ω' with compact closure in Ω, we say that f is *locally Hölder continuous* in Ω.

For $\beta \in (0, 2]$, we define

$$\langle f \rangle_{\beta,X_0} := \sup_{(x_0,t) \in \Omega \setminus \{x_0\}} \frac{\|f(x_0, t) - f(X_0)\|}{|t - t_0|^{\beta/2}}$$

and

$$\langle f \rangle_{\beta,\Omega} := \sup_{X_0 \in \Omega} \langle f \rangle_{\beta;X_0} .$$

For any $a > 0$, we write $a = k + \alpha$, where k is a nonnegative integer and $\alpha \in (0,1]$, and we define

$$\begin{aligned}
\langle f \rangle_{a;\Omega} &= \sum_{|b|+2j=k-1} \langle \mathrm{D}_x^b \mathrm{D}_t^j f \rangle_{\alpha+1} , \\
[f]_{\alpha,\Omega} &= \sum_{|b|+2j=k} [\mathrm{D}_x^b \mathrm{D}_t^j f]_\alpha , \\
|f|_{a,\Omega} &= \sum_{|b|+2j \le k} \sup_{\Omega} |\mathrm{D}_x^b \mathrm{D}_l^j f| + [f]_a + \langle f \rangle_a .
\end{aligned}$$

Then $|\cdot|_a$ defines a norm on $\mathscr{C}^{k+\alpha,\frac{\alpha}{2}}(\Omega) := \{f : |f|_{k+\alpha} < \infty\}$, which makes $\mathscr{C}^a(\Omega)$ a Banach space. We set $\mathscr{C}^{\alpha,\frac{\alpha}{2}}(\Omega) := \mathscr{C}^{0+\alpha,\frac{\alpha}{2}}(\Omega)$.

Bibliography

[AH11] Ben Andrews and Christopher Hopper. *The Ricci Flow in Riemannian Geometry: A Complete Proof of the Differentiable 1/4-Pinching Sphere Theorem*. Lecture Notes in Mathematics no. 2011. Springer-Verlag Berlin Heidelberg, 2011.

[AL86] Uwe Abresch and Joel Langer. *The normalized curve shortening flow and homothetic solutions*, *J. Differential Geom.* **23**, no. 2 (1986), pp. 175–196.

[BM12] Alexander A. Borisenko and Vicente Miquel. *Mean Curvature Flow of Graphs in Warped Products*, *Trans. Amer. Math. Soc.* **364**, no. 9 (2012), pp. 4551–4587.

[Bra78] Kenneth A. Brakke. *The Motion of a Surface by its Mean Curvature*. Mathematical Notes. Princeton University Press, 1978.

[CCH12] Albert Chau, Jingyi Chen, and Weiyong He. *Lagrangian mean curvature flow for entire Lipschitz graphs*, *Calc. Var. Partial Differential Equations* **44**, no. 1–2 (2012), pp. 199–220.

[CCY13] Albert Chau, Jingyi Chen, and Yu Yuan. *Lagrangian mean curvature flow for entire Lipschitz graphs II*, English, *Math. Ann.* **357**, no. 1 (2013), pp. 165–183.

[CLT02] JingYi Chen, Jia Yu Li, and Gang Tian. *Two-Dimensional Graphs Moving by Mean Curvature Flow*, English, *Acta Math. Sinica (N.S.)* **18**, no. 2 (2002), pp. 209–224.

[DeT83] Dennis M. DeTurck. *Deforming metrics in the direction of their Ricci tensors*, *J. Differential Geom.* **18**, no. 1 (1983), pp. 157–162.

[Eck04] Klaus Ecker. *Regularity Theory for Mean Curvature Flow*. Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser Boston, 2004.

[EH89] Klaus Ecker and Gerhard Huisken. *Mean curvature evolution of entire graphs*, *Ann. of Math. (2)* **130**, no. 3 (1989), pp. 453–471.

[EH90] Klaus Ecker and Gerhard Huisken. *A Bernstein result for minimal graphs of controlled growth*, *J. Differential Geom.* **31**, no. 2 (1990), pp. 397–400.

[EH91] Klaus Ecker and Gerhard Huisken. *Interior estimates for hypersurfaces moving by mean curvature*, English, *Invent. Math.* **105**, no. 1 (1991), pp. 547–569.

[Gag84] Michael E. Gage. *Curve shortening makes convex curves circular*, English, *Invent. Math.* **76**, no. 2 (1984), pp. 357–364.

[GH86] Michael E. Gage and Richard S. Hamilton. *The heat equation shrinking convex plane curves*, *J. Differential Geom.* **23**, no. 1 (1986), pp. 69–96.

[Gut13] Larry Guth. *Contraction of Areas vs. Topology of Mappings*, English, *Geom. Funct. Anal.* **23**, no. 6 (2013), pp. 1804–1902.

[Ham82] Richard S. Hamilton. *Three-manifolds with positive Ricci curvature*, *J. Differential Geom.* **17**, no. 2 (1982), pp. 255–306.

[Ham86] Richard S. Hamilton. *Four-manifolds with positive curvature operator*, *J. Differential Geom.* **24**, no. 2 (1986), pp. 153–179.

[Hui84] Gerhard Huisken. *Flow by mean Curvature of convex surfaces into spheres*, *J. Differential Geom.* **20** (1984), pp. 237–266.

[Hui90] Gerhard Huisken. *Asymptotic behavior for singularities of the mean curvature flow*, *J. Differential Geom.* **31**, no. 1 (1990), pp. 285–299.

[Ili61] Arlen M. Il′in. *On the behavior of the solution of the Cauchy problem for a parabolic equation under unrestricted growth of time*, *Uspekhi Mat. Nauk* **16**, no. 2 (1961). (Russian), pp. 115–121.

[Jos07] Jürgen Jost. *Partial Differential Equations*. **214**. Graduate Texts in Mathematics. 2nd Edition. Springer-Verlag, 2007.

[JX99] Jürgen Jost and Yuan-Long Xin. *Bernstein type theorems for higher codimension*, English, *Calc. Var. Partial Differential Equations* **9**, no. 4 (1999), pp. 277–296.

[JXY11] Jürgen Jost, Yuan-Long Xin, and Ling Yang. *The geometry of Grassmannian manifolds and Bernstein type theorems for higher codimension* (2011). arXiv:1109.6394 [math.DG].

[JXY12] Jürgen Jost, Yuan-Long Xin, and Ling Yang. *The regularity of harmonic maps into spheres and applications to Bernstein problems*, *J. Differential Geom.* **90**, no. 1 (2012), pp. 131–176.

[JXY13] Jürgen Jost, Yuan-Long Xin, and Ling Yang. *The Gauss image of entire graphs of higher codimension and Bernstein type theorems*, English, *Calc. Var. Partial Differential Equations* **47**, no. 3–4 (2013), pp. 711–737.

[Lie96] Gary M. Lieberman. *Second Order Parabolic Differential Equations*. 2005 Reprint. World Scientific Publishing Co. Pte. Ltd., 1996.

[LL11] Kuo-Wei Lee and Yng-Ing Lee. *Mean Curvature Flow of graphs of maps between Compact manifolds*, *Trans. Amer. Math. Soc.* **363**, no. 11 (2011), pp. 5745–5759.

[LL92] An-Min Li and Jimin Li. *An intrinsic rigidity theorem for minimal submanifolds in a sphere*, English, *Archiv der Mathematik* **58**, no. 6 (1992), pp. 582–594.

[LS11] Guanghan Li and Isabel M. C. Salavessa. *Mean curvature flow of spacelike graphs*, English, *Math. Z.* **269**, no. 3–4 (2011), pp. 697–719.

[Man11] Carlo Mantegazza. *Lecture Notes on Mean Curvature Flow*. **290**. Progress in Mathematics no. 12. Birkhäuser Basel, 2011.

[Mul56] William W. Mullins. *Two-dimensional Motion of Idealized Grain Boundaries*, *J. Appl. Phys.* **27**, no. 8 (1956), pp. 900–904.

[MW11] Ivana Medoš and Mu-Tao Wang. *Deforming symplectomorphisms of complex projective spaces by the mean curvature flow*, *J. Differential Geom.* **87**, no. 2 (2011), pp. 309–342.

[Nas56] John Nash. *The imbedding problem for Riemannian manifolds*, *Ann. Math.* **63**, no. 1 (1956), pp. 20–63.

[Smo04] Knut Smoczyk. *Longtime existence of the Lagrangian mean curvature flow*, English, *Calc. Var. Partial Differential Equations* **20**, no. 1 (2004), pp. 25–46.

[Smo12] Knut Smoczyk. *Mean Curvature Flow in Higher Codimension: Introduction and Survey*. In: Christian Bär, Joachim Lohkamp, and Matthias Schwarz, eds., *Global Differential Geometry*, **17**, Springer Proceedings in Mathematics, Springer Berlin Heidelberg, 2012, pp. 231–274.

[Smo96] Knut Smoczyk. *A canonical way to deform a Lagrangian submanifold* (1996). arXiv:dg-ga/9605005.

[SS14a] Andreas Savas-Halilaj and Knut Smoczyk. *Bernstein Theorems for length and area decreasing Minimal Maps*, *Calc. Var. Partial Differential Equations* **50**, no. 3–4 (2014), pp. 549–577.

[SS14b] Andreas Savas-Halilaj and Knut Smoczyk. *Evolution of contractions by mean curvature flow*, English, *Math. Ann.* **361**, no. 3–4 (2014), pp. 725–740.

[SS14c] Andreas Savas-Halilaj and Knut Smoczyk. *Homotopy of area decreasing maps by mean curvature flow*, *Adv. Math.* **255** (2014), pp. 455–473.

[STW14] Knut Smoczyk, Mao-Pei Tsui, and Mu-Tao Wang. *Curvature Decay Estimates of Graphical Mean Curvature Flow in Higher Codimensions* (2014). arXiv:1401.4154 [math.DG].

[TW04] Mao-Pei Tsui and Mu-Tao Wang. *Mean curvature flows and isotopy of maps between spheres*, *Comm. Pure Appl. Math.* **57**, no. 8 (2004), pp. 1110–1126.

[Wan01] Mu-Tao Wang. *Deforming area preserving diffeomorphism of surfaces by mean Curvature flow*, *Math. Res. Lett.* **8**, no. 5 (2001), pp. 651–661.

[Wan02] Mu-Tao Wang. *Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension*, English, *Invent. Math.* **148**, no. 3 (2002), pp. 525–543.

[Wan03] Mu-Tao Wang. *On graphic Bernstein type results in higher codimension*, *Trans. Amer. Math. Soc.* **355** (2003), pp. 265–271.

[Wan05] Mu-Tao Wang. *Subsets of Grassmannians Preserved by Mean Curvature Flow*, *Comm. Anal. Geom.* **13**, no. 5 (2005), pp. 981–998.

[Wan08] Mu-Tao Wang. *Lectures on mean curvature flows in higher codimensions*. In: Lizhen Ji et al., eds., *Handbook of geometric analysis*, **7**, Advanced Lectures in Mathematics (ALM) no. 1, International Press of Boston, Inc., 2008, pp. 525–543, arXiv:1104:3354 [math.DG].

[Whi05] Brian White. *A Local Regularity Theorem for Mean Curvature Flow*, *Ann. Math.* **161**, no. 3 (2005), pp. 1487–1519.

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Publications

[2] **F. Lubbe** and L. Schäfer, *Pseudo-holomorphic curves in nearly Kähler manifolds*, Differ. Geom. Appl. **36** (2014), 24–43.

[1] I. Bauer, T. A. Ivanova, O. Lechtenfeld and **F. Lubbe**, *Yang-Mills instantons and dyons on homogeneous G_2-manifolds*, JHEP **10** (2010) 044, `arXiv:1006.2388`.

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